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A Measure-Valued Approach to Convex Set-Valued Dynamics

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Abstract. In this paper we introduce a new concept of differentiable set-valued dynamics and show the existence of solution curves. To do this we introduce simple set-valued functions which are completely determined by a convex set and a function on the boundary of this set, and show the differentiability of these functions from the right at $t = 0$. Then we approximate local solutions of our dynamical system by set-valued functions consisting of a finite number of simple functions. The dynamics are described by functions on the boundaries of convex bodies. (This is analogous to the description of single-valued dynamics by vector fields.) It can be shown that in some special cases applicable to set-valued constraint stochastic optimization, the local solutions can be glued together to provide global ones. A constraint stochastic optimization problem is investigated.

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1. Introduction

This paper is concerned with new set-valued differentiable dynamics. The concept of differentiability (essential for differentiable dynamics) means that we can approximate certain mathematical objects (differentiable single or set valued functions) in a neighborhood of a point x by much simpler objects up to order one. In any situation one deals with differentiability one has to specify the class of differentiable objects. In most cases, this is done by specifying the class of simpler first-order approximates first. In the case of single-valued differentiation the first-order approximates are the affine functions and the class of differentiable functions is defined as the class of all functions which can be approximated up to order one by affine ones. So we have to look for substitutes of affine mappings in set-valued differentiation.

The most common substitutes are the various kinds of tangent cones which provide the right framework for a differentiability theory for set-valued functions which is applicable to differential inclusions. (For an overview of this approach to set-valued differentiation see [2, 4, 14] and the literature cited there. For an overview on the subject of differential inclusions see [5] and [9] and for the subject of viability [3].)

But there are several other approaches to set-valued differentiation which all start with the definition of first-order approximates.

In an approach of Artstein [1] the first-order approximation is done by multiaffines. The graph of a multiaffine mapping is the union of the graphs of affine mappings.

In the approach of Silin [15] the first-order approximates are called quasiaffine mappings. A quasiaffine mapping is a convex set-valued mapping which is obtained as the intersection of half space valued mappings, each half space moving with constant velocity.

In the approach of Aubin [2] the approximation is done by transitions. The concept of a transition is a very general axiomatic one; it frees the concept of differentiation from the concept of linear structure. (This approach provides a meta theory to the various concepts of set valued analysis and might hopefully be the starting point for a unified abstract theory of set-valued analysis.)

We call the substitutes for affine mappings introduced by this paper simple (convex-valued) functions. But the differentiability structure in our approach is not obtained from these simple functions. It is obtained from the space of Borel measures on $\mathbb{E}^k$. Thus we have to prove differentiability of simple functions to show that they can serve as more 'simple' first-order approximates.

In the case of single valued dynamical systems the operation of integration, which is inverse to differentiation, is introduced next. This is done since it provides a tool for obtaining the solution of the differentiable dynamics. In the set-valued case integration is in general not inverse to differentiation. (For an approach where it is reverse see [15]; for more common concepts, see [14] and [4].)

In this paper, however, we are not concerned with the problem of integration. Integration is inverse to differentiation in our approach, in most cases it is trivial (see below) and of no relevance for the problems considered here.

We present here a new approach for the evolution of convex set-valued mappings. This approach is analogous to single-valued gradient dynamics and can be used to solve constraint stochastic optimization problems. The application to stochastic optimization will be presented in another paper. (To get an idea how our concepts can be applied to stochastic optimization, see Example 7.)

What mainly distinguishes our approach to set valued analysis from all others (except the abstract approach of mutational analysis) is that set valued differentiability in our approach is not directly connected to the vector space structure of $\mathbb{E}^k$. It is neither connected by vector fields or tangent cones, nor is it connected to the linear structure of $\mathbb{E}^k$ by affine mappings, families of affine mappings or half spaces. Although our approach inherits its differentiability structure from the vector space of Borel measures on $\mathbb{E}^k$ (see below), it does not inherit any algebraic structure from this space. (This is contrasted by the approach presented in [6], which uses the fact that the space of nonempty compact convex sets [endowed with the Hausdorff metric and Minkowski addition] is metric and algebraic isomorphic with a subset of a certain Banach space, to establish results on set valued differential equations.)

So in the approach presented here vector space structure is only used in the geometric arguments of the proofs and since the properties of Lebesgue measure and mixed volumes strongly rely on the special structure of $\mathbb{E}^k$. But the definition of differentiability as well as the method of constructing simple functions (except the aspect of convexity) can be extended to arbitrary locally compact metric spaces.

Our approach is also essentially different from the methods of shape sensitivity analysis presented in [16]. In shape sensitivity analysis more general set functionals (domain functionals) than non negative measures are considered (most of them related to partial differential equations (see [16], Section 2.5)).

Shape sensitivity analysis considers the change of a domain (set) with respect to a transformation ([16], Section 2.8) obtained from a vector field ([16], Section 2.9), whereas in our approach the change is given by the addition and subtraction of balls. In shape sensitivity analysis the derivative of a domain functional is determined by a distribution on the boundary of the domain which acts on the space of vector fields on $\mathbb{E}^k$ ([10], Theorem 3.2). In case of a C^k boundary $k \geqslant 1$ this distribution is equivalent to a scalar distribution on the boundary of the domain (see [10], Corollary 2, or [16], Theorem 2.27). This is similar to our approach where the volume differential (defined at the begin of Section 3) is determined by a measure on the boundary of the set. The approach developed in this paper does not rely on the differentiability of the boundary of the sets under consideration right from the beginning, since it does not explicitly use concepts like vector fields orthogonal to boundaries. (See [16], Theorem 2.27 and [10], Corollary 2 for the importance of smooth boundaries in shape sensitivity analysis; but see also Example 1 and the Propositions 1 and 2 below for the implicit use of normal vectors in our approach.)

What is also crucial and different from many other problems in set-valued analysis is that by the correction function κ that is necessary for solving constraint optimization problems (as presented in Example 7) the evolution of our convex valued solution $C(t)$ also locally depends on the whole set $C(t)$ (for a general abstract approach to such situation see [2], Chapter 4); i.e. it is in general not sufficient to consider a neighborhood $U(x)$ of a point $x \in \partial C(t)$ to determine the local change around x of $\partial C(t)$ with t. In this it differs from the solution set of a differential inclusion as well as from the evolution of sets considered in shape sensitivity analysis. (If $x(.)$ is a solution of a differential inclusion $x(t) \in F(t, x(t))$ then the change of $x(.)$ at the point $(x(t))$ only depends on $x(t)$ and t and not on the state $y(t)$ of any another solution $y(.)$. For investigations on the structure of the solution sets of differential inclusions see [8].)

We start by describing our approach for the special case of Lebesgue measure λ and by introducing our concept of differentiability:

In the following we denote the space of convex compact subsets of $\mathbb{E}^k$ endowed with the Hausdorff metric d by $\mathcal{C}(\mathbb{E}^k)$ respectively by $(\mathcal{C}, d)$ or simply by $\mathcal{C}$. We also denote the convex hull of an arbitrary set $S \subset \mathbb{E}^k$ by $\operatorname{conv}(S)$ and the convex hull of two points $a, b \in \mathbb{E}^k$ by $[a, b]$. The diameter of a set S is denoted $\operatorname{diam}(S)$, the interior of an arbitrary set S by $\operatorname{int}(S)$ and the interior of the unit ball $\mathbb{B} \subset \mathbb{E}^k$ by $\mathbb{B}^\circ$. The power set of $\mathbb{E}^k$ is denoted by $2^{\mathbb{E}^k}$. Given a function f we denote by $f|_S$ the restriction of f to the set S. We say that a function $t \mapsto C(t)$ is convex-valued or convex set-valued if it is a function from some interval $[0, T]$, $T \in (0, \infty]$ to the space $(\mathcal{C}, d)$. By ∂C we denote the boundary of a set C. We denote by $\|.\|$ the Euclidean norm on $\mathbb{E}^k$, by $\mathbb{1}_S(.)$ the characteristic function of a set $S \subseteq \mathbb{E}^k$ and by $\operatorname{sgn}(.)$ the signum function on $\mathbb{R}$. Given two sets $S, S' \subseteq E^d$ we define the Minkowski sum by $S + S' := \bigcup_{x \in S'}(S + x)$ and the Minkowski difference by $S \ominus S' := \bigcap_{x \in S'}(S - x)$. With aff we denote the affine hull.

We embed the space $\mathcal{C}(\mathbb{E}^k)$ into the space of Borel measures on $\mathbb{E}^k$ by the embedding

$$C \mapsto \lambda|_C, \quad \text{where } \lambda|_C(A) := \lambda(A \cap C)$$

and λ denotes Lebesgue measure. Equivalently, we identify a convex compact set C with the operator

$$\psi \mapsto \int_C \psi \, \mathrm{d}\lambda = \int \psi \, \mathrm{d}\lambda|_C$$

on the space of continuous functions.

In any case, we inherit the weak differentiability structure from the space of Borel measures. To be more precise, we call a set-valued function $C(.)$ with compact convex values weakly differentiable from the right if $\lambda|_{C(.)}$ is weakly differentiable from the right; i.e. if there exists a measure $\dot{C}^\lambda(t)$ such that

$$\lim_{h \to 0} \int \psi \frac{\mathrm{d}\lambda|_{C(t+h)} - \mathrm{d}\lambda|_{C(t)}}{h} = \int \psi \, \mathrm{d}\dot{C}^\lambda(t)$$

for any continuous function ψ. This concept of weak differentiability of set-valued functions is due to G. Pflug ([12], Section 3.2.2, pp. 168 ff).

In this paper we only consider such differentiable set-valued mappings $C(.)$ with a derivative $\dot{C}^\lambda(t)$ which is absolutely continuous with respect to the surface area measure $o_{\partial C(t)}$ of $\partial C(t)$. (An example of a differentiable set valued mapping with derivative not absolutely continuous with respect to $o_{\partial C(t)}$ is given in Example 11 at the end of the Introduction.) Except for Example 11, the term differentiable set-valued function will denote a set-valued function weakly differentiable from the right, with derivative absolutely continuous with respect to $o_{\partial C(t)}$.

So given a convex-valued differentiable mapping $C(.)$ we can see that the density-function $f(., .)$

$$f(t, x)|_{\partial C(t)} = \frac{\mathrm{d}\dot{C}^\lambda(t)}{\mathrm{d}o_{\partial C(t)}}$$

completely describes the derivative $\dot{C}^\lambda(t)$ of the function $C(.)$.

Before we come to our first example of a family of set-valued differentiable mappings, we will remark the following:

The set of differentiability points $\operatorname{diff}(\partial C)$ on the boundary ∂C of a convex set $C \subset \mathbb{E}^k$ (i.e. the set of points $x \in \partial C$ with unique tangent hyperplane) is of full measure with respect to the surface area measure $o_{\partial C}$ of ∂C. This is a consequence of the Rademacher Theorem (see [13]).

Therefore a measure $\dot{C}^\lambda$ absolutely continuous with respect to $o_{\partial C}$ is uniquely determined by the restriction of the Radon–Nykodym derivative $\mathrm{d}\dot{C}^\lambda(t)/\mathrm{d}o_{\partial C}$ to the set $\operatorname{diff}(\partial C)$ of differentiability points of C.

EXAMPLE 1 (Moving convex bodies). Let $C \subseteq \mathbb{E}^k$ be a convex body (i.e. a convex compact set with nonempty interior). Further, let $\gamma\colon [0, T] \mapsto \mathbb{E}^k$ be a curve which is differentiable from the right. Let $C(t) = C + \gamma(t)$ for $t \in [0, T]$. Then the set-valued mapping $t \mapsto C(t)$ is weakly differentiable from the right and its derivative $\dot{C}^\lambda(t)$ (with respect to Lebesgue measure λ) is given by

$$\frac{\mathrm{d}\dot{C}^\lambda(t)}{\mathrm{d}o_{\partial C(t)}}(x) = \langle \eta(x, t), \gamma(t) \rangle$$

for all $x \in \operatorname{diff}(\partial C(t))$ and $\eta(x, t)$, the outward unit normal vector of $\partial C(t)$ at x. By the remark above we can see that $\dot{C}^\lambda(t)$ is uniquely determined by this formula. The result is easily proved for polyhedra and is extended to the case of general convex bodies by an approximation argument. We omit the prove but mention the following important special case explicitly:

EXAMPLE 2 (The moving ball). Let $C(t) \subset \mathbb{E}^k$ be given by $C(t) = r\mathbb{B} + (r \cdot t, 0, \ldots, 0)$ where $\mathbb{B}$ denotes the closed Euclidean unit ball; i.e. the mapping $t \mapsto C(t)$ describes the ball of radius r moving with velocity r along the x_1-axes. The derivative of this mapping is given by

$$\frac{\mathrm{d}\dot{C}^\lambda(t)}{\mathrm{d}o_{\partial C(t)}}(x_1, \ldots, x_k) = x_1 - r \cdot t \quad \text{for } (x_1, \ldots, x_k) \in \partial C(t).$$

More general, we not only consider moving balls but also balls which move and expand at the same time:

EXAMPLE 3 (Moving and expanding balls). Let $C(t) \subset \mathbb{E}^k$ be given by $C(t) = r(t)\mathbb{B} + (\int_0^t r(s)\,\mathrm{d}s, 0, \ldots, 0)$, where $r\colon [0, T] \mapsto [0, \infty)$ denotes a function differentiable from the right with the right-hand side derivative $r_+(t)$. Then the derivative $\dot{C}^\lambda(t)$ of $C(t)$ is given by

$$\frac{\mathrm{d}\dot{C}^\lambda}{\mathrm{d}o_{\partial C(t)}}(x_1, \ldots, x_k) = x_1 - \int_0^t r(s)\,\mathrm{d}s + r_+(t) \quad \text{for } (x_1, \ldots, x_k) \in \partial C(t).$$

EXAMPLE 4 (Rotating convex sets in $\mathbb{E}^2$). Suppose that $t \mapsto C(t)$ is the set-valued mapping which describes the rotation of a convex body $C \subset \mathbb{E}^2$, which rotates with velocity 1 in counterclockwise direction around the origin. Let $\gamma_t\colon [0, s) \mapsto \partial C(t)$ be a continuous bijection which maps $[0, s)$ in counterclockwise direction onto $\partial C(t)$. Let further γ be such that if we let

$$\gamma_+(s') = \lim_{s'' \to s'+} \frac{\gamma(s' - s'')}{s' - s''},$$

then $\gamma_+(s')$ exists and $|\gamma_+(s')| = 1$ for all $s' \in [0, s)$. Then the weak derivative (with respect to Lebesgue measure λ) is given by

$$\frac{\mathrm{d}\dot{C}^\lambda(t)}{\mathrm{d}o_{\partial C(t)}}(x) = -\langle x, \gamma_+(\gamma^{-1}(x))\rangle.$$

After having defined the differentiation procedure and seen some examples, we are of course interested in the reverse of this procedure. We saw in all the examples above, that the derivative of the mapping $t \mapsto C(t)$ is a mapping $t \mapsto \dot{C}^\lambda(t)$ with $\dot{C}^\lambda(t)$ a signed measures with support $\partial C(t)$. (Note that in general the support $\mathrm{supp}(\dot{C}^\lambda(t))$ of $\dot{C}^\lambda(t)$ is only a subset of $\partial C(t)$.) In the case $\mathrm{supp}(\dot{C}^\lambda(t)) = \partial C(t)$ the reverse operation of integration becomes trivial, since we have $C(t) = \mathrm{conv}(\mathrm{supp}(\dot{C}^\lambda(t)))$. We will not develop the concept of integration further, since it is of no relevance for the problems considered in this article and the trivial case $C(t) = \mathrm{conv}(\mathrm{supp}(\dot{C}^\lambda(t)))$ is rather typical.

What is not trivial is to consider set-valued differential equations of the form

$$\dot{C}^\lambda(t) = F(C(t), t), \qquad C(0) = C_0$$

with F a mapping from $\mathcal{C} \times [0, T]$ to the space of signed measures on $\mathbb{E}^k$ and C_0 a convex set.

We consider in this paper the spacial case, that

$$F(C(t), t) = f(t, x)|_{\partial C(t)}\, \mathrm{d}o_{\partial C(t)}$$

for some given function $f\colon \mathbb{R} \times \mathbb{E}^k \mapsto \mathbb{R}$.

So we are interested in the following problem:

(1) Given a set $C_0 \in \mathcal{C}(\mathbb{E}^k)$ and a function $f\colon \mathbb{R} \times \mathbb{E}^k \mapsto \mathbb{R}$, does there exist a $\rho \in (0, T]$ and a function $C\colon [0, \rho) \mapsto \mathcal{C}(\mathbb{E}^k)$ such that

$$\frac{\mathrm{d}\dot{C}^\lambda(t)}{\mathrm{d}o_{\partial C(t)}} = f(t, x)|_{\partial C(t)} \quad \text{and} \quad C(0) = C_0;$$

i.e. does there exist a set valued function $C(.)$ which follows the dynamics given by the restriction of functions to the boundary ∂C of the sets $C \in \mathcal{C}(\mathbb{E}^k)$?

Given a measure μ with a continuous density g with respect to Lebesgue measure λ we can consider weak differentiation with respect to μ and can either define the derivative $\dot{C}^{\mu}(t)$ analogous to $\dot{C}^{\lambda}(t)$ defined above, or equivalently define $\dot{C}^{\mu}(t)$ by

$$\dot{C}^{\mu}(t) := g|_{\partial C(t)}\dot{C}^{\lambda}(t)$$

if $\dot{C}^{\lambda}$ exists. This leads to a slightly different formulation of problem (1) as follows:

(2) Given $C_0 \in \mathcal{C}(\mathbb{E}^k)$ and $f\colon \mathbb{R} \times \mathbb{E}^k \mapsto \mathbb{R}$ find a ρ and a set valued function $C(.)$ on $[0, \rho)$ such that

$$\frac{\mathrm{d}\dot{C}^{\mu}(t)}{\mathrm{d}o_{\partial C(t)}} = f(t, x)|_{\partial C(t)} \cdot g(x)|_{\partial C(t)} \quad \text{and} \quad C(0) = C_0.$$

We solve this problem for μ with Lipschitz continuous density g in case that $f(.,.)$ is Lipschitz continuous, that $f(t, .)$ is concave for any $t \in [0, T]$ and that

$$\text{(3)} \qquad \int_{\partial C(t)} f(t, .) \cdot g(.)\, \mathrm{d}o_{\partial C(t)} = 0.$$

The solution $C(t)$ fulfills $\mu(C(t)) = \mu(C_0)$.

EXAMPLE 5 (The moving ball revisited). If we let $C(t)$ be the set-valued mapping of Example 2, $f(t, x) = x_1 - r \cdot t$ and consider the case $g = 1$ (i.e., $\mu = \lambda$), then we observe that Equation (3) is fulfilled and that $\lambda(C(t)) = \lambda(C_0)$.

More general, we show in Theorem 1 that if $f\colon \mathbb{R} \times \mathbb{E}^k \mapsto \mathbb{R}$ is a Lipschitz continuous function, for that $f(t, .)$ is convex for any $t \in \mathbb{R}$ and if g is a Lipschitz continuous density and $C_0 \in \mathcal{C}(\mathbb{E}^k)$, then there exists a ρ and a correction function $\kappa\colon [0, \rho) \mapsto \mathbb{R}$ such that for all $t \in [0, \rho)$ we have

$$\frac{\mathrm{d}\dot{C}^{\mu(t)}}{\mathrm{d}o_{\partial C(t)}}(x) = [f(t, x) - \kappa(t)] \cdot g(x) \tag{4}$$

and

$$\int_{\partial C(t)} [f(t, .) - \kappa(t)] \cdot g(.)\, \mathrm{d}o_{\partial C(t)} = 0. \tag{5}$$

We further show that again $\mu(C(t)) = \mu(C_0)$ holds. This is caused by the fact that the volume correction function $\kappa(.)$ has to fulfill Equation (5). Together Equations (4) and (5) imply that the infinitesimal change of $\mu(C(t))$ equals 0 for all t and thus that $\mu(C(t))$ does not change at all.

EXAMPLE 6 (Moving and expanding balls and the correction function $\kappa(t)$). Let C_0 be a ball of radius r_0 centered in the origin, let μ be a measure with

Lipschitz continuous density $g > 0$ a.e. with respect to Lebesgue measure and let $f\colon \mathbb{R} \times \mathbb{E}^k$ be given by $f(t, x_1, \ldots, x_k) = x_1$. Then there exists a correction function $\kappa\colon [0, \infty) \mapsto \mathbb{R}$ and a set valued function $C(.)$, which fulfill (4) and (5) on $[0, \infty)$ such that $C(.)$ is of the form described in Example 3; i.e. $C(t) = r(t)\mathbb{B} + (\int_0^t r(s)\, \mathrm{d}s, \vec{0})$. (A proof of this statement is sketched in the Appendix.)

The local solutions $C\colon [0, \rho) \mapsto \mathcal{C}(\mathbb{E}^k)$ (provided by Theorem 1) can be glued together to give a global solution $C(t)$ such that if $\lim_{t\to\infty} C(t)$ exists then $D = \lim_{t\to\infty} C(t)$ maximizes $\int_D f\, \mathrm{d}\mu$ with respect to the constraint $\mu(D) = \mu(C_0)$. This dynamically solves the problem of optimizing $\nu(C)$ with respect to a constraint $\mu(C) = \alpha$ in case there exists a monotonic function h such that $h \circ \mathrm{d}\nu/\mathrm{d}\mu$ is concave; i.e. if we set $f := h \circ \mathrm{d}\nu/\mathrm{d}\mu$, then the limit $D := \lim_{t\to\infty} C(t)$ of a global solution of our system of Equations (4) and (5) with suitable correction function κ solves

$$\begin{aligned} &\text{maximize} \quad \nu(C) \\ &\text{subject to} \quad \mu(C) = \mu(C_0). \end{aligned}$$

EXAMPLE 7 (Optimal statistical tests). Let $\alpha > 0$, let

$$\mu_\alpha := N\left(\begin{pmatrix} -\alpha \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

and let

$$\nu := N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Then for any $\alpha \in (0, \infty)$ there exists a strictly monotone increasing function $h_\alpha\colon [0, \infty) \mapsto \mathbb{R}$, such that $h_\alpha \circ \mathrm{d}\nu/\mathrm{d}\mu_\alpha(x_1, x_2) = x_1$. So if we let $C_0 := r_0\mathbb{B} \subset \mathbb{E}^2$ for some $r_0 > 0$, then we are in the situation of Example 6. Thus there exists for any $\alpha > 0$ a global solution $C_\alpha(t)$ of the Equations (4) and (5) so that $C_\alpha(t)$ is a ball for any $t \in [0, \infty)$. Since the μ_α are probability measures, the limit $C_\alpha(\infty) := \lim_{t\mapsto\infty} C_\alpha(t)$ exists (in the sense of Painlevé–Kuratowski) and by the special structure of the solution $C_\alpha(\infty)$ is a half space. We further note that $C_\alpha(\infty)$ solves the optimization problem

$$\begin{aligned} &\text{maximize} \quad \nu(C), \\ &\text{subject to} \quad \mu_\alpha(C) = \mu_\alpha(C_0). \end{aligned}$$

Thus $1_{C_\alpha(\infty)}$ is a most powerful statistical test for testing the hypothesis μ_α against the hypothesis ν. The level of significance of the test equals $\mu(C_\alpha(\infty)) = \mu(C_0)$ and is thus determined by the choice of the radius of the starting circle C_0.

We also note that our dynamics are a natural generalization of the dynamics obtained from the gradient field of a concave function to the set-valued case. That gradient dynamics are indeed a limit case of our approach is indicated by the following extension of Example 2:

EXAMPLE 8 (Gradient dynamics in the limit). Let $f(t, x_1, \dots, x_k) = x_1 =: \hat{f}(x_1, \dots, x_k)$ and $C_0(r) = r\mathbb{B}$. Then by Example 2 we have that $C(r)(t) = r\mathbb{B} + (r \cdot t, 0, \dots, 0)$ fulfills the equation

$$\frac{\mathrm{d}\dot{C}^{\lambda}(r)(t)}{\mathrm{d}o_{\partial C(r)(t)}} = \hat{f}(x_1, \dots, x_k)|_{\partial C(r)(t) - r\cdot t}.$$

Let $\xi(t) = \lim_{r\to 0} C(r)(\frac{t}{r})$, then $\xi(t)$ fulfills

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t} = (1, 0, \dots, 0) = \nabla \hat{f}(x_1, \dots, x_k).$$

Global solutions (as presented in Examples 6 and 7) will be discussed in a more general context in another paper. Here we confine ourselves to proving the existence of local solutions.

Note that the solutions $C(t)$ of (1) and (2) are in general not uniquely determined by C_0 and $f(.,.)$ (Of course, the same is true for the solutions $C(t)$ and $\kappa(t)$ of the system of Equations (4) and (5).) We give the following very simple example:

EXAMPLE 9 (Solutions are not unique). Let C_0 be the unit square centered at $(0, 0)$ with one vertex at $(-1, -1)$. Let $f\colon \mathbb{R} \times \mathbb{E}^k \mapsto \mathbb{R}$ be the constant function $f(t, x) = 1$ and let

$$C(s)(t) = \begin{cases} (1+t)C_0 & \text{if } 0 \leqslant t \leqslant s\,, \\ (1+s)C_0 + (t-s)\mathbb{B} & \text{if } 0 \leqslant s < t\,. \end{cases}$$

Then for any $s \in [0, \infty)$ the set valued function $C(s)(.)$ is a solution of

$$\frac{\mathrm{d}\dot{C}(s)^{\lambda}(t)}{\mathrm{d}o_{\partial C(s)(t)}} = \mathbb{1}_{\partial C(s)(t)} \quad \text{with } C(s)(0) = C_0.$$

So we see that even in the simplest case of $f(.,.) = 1$ the solution of Equation (1) is not unique. This fact is related to the nonsmoothness of the boundary of the sets $C(s)(t)$ for $t \leqslant s$ which arises from the nonsmoothness of the set C_0. But even if the set C_0 is infinitely smooth, a solution of (1) will in general not consist of smooth sets only as the following example shows:

EXAMPLE 10. Let D denote the square centered at $(0, 0)$ with one vertex at $(-1, -1)$ and let $C_0 = D + \mathbb{B}$. Let $f(.,.) = -1$, then the set valued mapping $C\colon [0, \infty) \mapsto \mathcal{C}(\mathbb{E}^k)$ given by

$$C(t) = \begin{cases} D + (1-t)\mathbb{B} & \text{for } 0 \leqslant t < 1, \\ (2-t)D & \text{for } 1 \leqslant t < 2, \\ \emptyset & \text{for } 2 \leqslant t, \end{cases}$$

is a solution of

$$\frac{\mathrm{d}\dot{C}^{\lambda}(t)}{\mathrm{d}o_{\partial C(t)}} = -\mathbb{1}_{\partial C(t)} \quad \text{with } C(0) = C_0 = D + \mathbb{B}.$$

For $1 \leqslant t < 2$ the convex sets $C(t)$ are not smooth.

We now come to the last example of the introduction which shows that the measures $\dot{C}^\lambda(t)$ are in general not absolutely continuous with respect to $o_{\partial C(t)}$.

EXAMPLE 11. Let $\mathbb{B} \subset \mathbb{E}^2$ and let $\theta(t)$: $[0, \infty) \mapsto [1, \infty)$ be the continuous function implicitly defined by

$$\lambda(\mathrm{conv}(\mathbb{B} + (\theta(t), 0)) \setminus \mathbb{B}) = t.$$

Then $C(t) := \mathrm{conv}(\mathbb{B} + (\theta(t), 0))$ is a differentiable convex function on $[0, \infty)$ whose derivative at $t = 0$ is given by $\dot{C}^\lambda(0) = \delta_{(1,0)}$ with $\delta_{(1,0)}$, the Dirac measure at the point $(1, 0) \in \mathbb{E}^2$.

The paper is organized as follows: In the following section (Section 2) we introduce simple convex-valued functions $C_f(t)$ (Lemma 1) which grow or shrink at any point with constant velocity and play the role of affine functions in single-valued analysis.

In Section 3 we show that $C_f(t)$ fulfills

$$\lim_{t\to 0} \int \psi \frac{\mathrm{d}\mu|_{C_f(t)} - \mathrm{d}\mu|_{C(0)}}{t} = \int_{\partial C(t)} \psi \cdot f \cdot g \, \mathrm{d}o,$$

which means that $C_f(t)$ is weakly differentiable from the right at $t = 0$ with derivative $f \cdot g \, \mathrm{d}o_{\partial C(t)}$ (Lemma 2). To do this we make use of the concept of mixed volumes (see [17, 11] and the Appendix).

In the final Section 4, we locally approximate a solution of the differential equation by set-valued functions consisting piecewise of simple functions analogous to the polygonal approximation in the single-valued case (Theorem 1; for numerical estimates on Euler approximations of set-valued differential equations in a different context, see [7]).

The Appendix collects mainly material on mixed volumes and two lemmas on the continuity of the volume differential $V'(C, f, g)$. For a detailed discussion of mixed volumes, consult [17] or [11].

2. Simple Convex-Valued Functions

To approximate the solution of a convex-valued differential equation of the form

$$\frac{\mathrm{d}\dot{C}^\lambda(t)}{\mathrm{d}o_{\partial C(t)}} = f(x)|_{\partial C(t)}, \qquad C(0) = C_0$$

up to order one locally for given $C_0 \in \mathcal{C}(\mathbb{E}^d)$, we need functions $C_f(t)$ such that

$$C_f(0) = C_0, \qquad \left.\frac{\mathrm{d}\dot{C}^\lambda(t)}{\mathrm{d}o_{\partial C(t)}}\right|_{t=0} = f(x)|_{\partial C_0}$$

and C_f is a convex-valued function. In the following lemma we define such functions C_f and show that they are convex-valued. The proof of the differentiability property of C_f is treated in the next section.

We say that a function f is concave, if it is real valued, defined on a convex set $C \subset E^d$ and for all $x_1, x_2 \in C$ and $\xi \in [0, 1]$ we have $\xi f(x_1) + (1 - \xi) f(x_2) \leqslant f(\xi x_1 + (1 - \xi)x_2)$.

LEMMA 1. *Let $C_0 \subset \mathbb{E}^k$ be a convex body (i.e. a compact convex set with non-empty interior) and $f\colon C_0 \mapsto \mathbb{R}$ be a concave Lipschitz continuous function on C_0 and $l \in \mathbb{R}$ a Lipschitz constant for f. Then the values of the set-valued mapping $C_f\colon [0, \frac{1}{l}] \mapsto 2^{\mathbb{E}^k}$ defined by*

$$C_f(t) := \left[C_0 \cup \bigcup_{\{x \in \partial C_0 | f(x) \geqslant 0\}} (x + tf(x)\mathbb{B}) \right] \Big\backslash \bigcup_{\{x \in \partial C_0 | f(x) < 0\}} (x + tf(x)\mathbb{B}^\circ)$$

are convex compact sets. We call the functions $t \mapsto C_f(t)$ simple convex-valued functions.

Remark. The easiest example of a simple convex-valued function is given by $C_f(t) := (1 + t) \cdot \mathbb{B}$. This function is obtained from $C_0 = \mathbb{B}$ and $f = 1$. It is the function which blows up the unit ball $\mathbb{B}$ with constant velocity 1. Note that the set-valued function $t \mapsto C(t)$ of Example 2 which describes a moving ball is not a simple convex-valued function. Another example of a simple function is given in Example 10. A more interesting example is provided by:

EXAMPLE 12 . Let

$$f(x_1, \ldots, x_k) = x_1 \quad \text{and} \quad C_0 = \operatorname{conv}\{(0, 1), (0, -1), (-1, 0)\} \subset \mathbb{E}^2,$$

then

$$[0, 1] \ni t \mapsto C_f(t) = \operatorname{conv}\left\{ (0, 1), (0, -1), \left(\lim_{\tilde{t} \to t-} \frac{1 - \tilde{t}\sqrt{2 - \tilde{t}^2}}{\tilde{t}^2 - 1}, 0 \right) \right\}$$

is the simple convex valued function corresponding to f and C_0.

Proof of Lemma 1. By continuity and boundedness of f it is clear that the sets $C_f(t)$ are compact. For $t \in [0, \frac{1}{l}]$ we set

$$C_f^+(t) := \bigcup_{\{x \in \partial C_0 | f(x) \geqslant 0\}} (x + tf(x)\mathbb{B})$$

and

$$C_f^-(t) := \bigcup_{\{x \in \partial C_0 | f(x) < 0\}} (x + tf(x)\mathbb{B}^\circ).$$

We prove that

(1) $C_f^+(t) \cap C_f^-(t) = \emptyset$,
(2) $C_0 \setminus C_f^-(t)$ is convex,
(3) $\text{conv}(C_f^+(t)) \subseteq [C_f^+(t) \cup C_0] \setminus C_f^-(t)$,
(4) If $y_1 \in C_f^+(t) \setminus C_0$ and $y_2 \in C_0 \setminus C_f^-(t)$ then for $\xi \in [0, 1]$ we have $\xi y_1 + (1-\xi)y_2 \in [C_f^+(t) \cup C_0] \setminus C_f^-(t)$.

Proof of (1). We argue indirectly: Assume that $z \in C_f^+(t) \cap C_f^-(t)$, then there exist x^+ and x^- such that $f(x^+) \geqslant 0$, $f(x^-) < 0$ and $z \in (x^+ + tf(x^+)\mathbb{B}) \cap (x^- + tf(x^-)\mathbb{B}^\circ)$. Thus we have

$$\begin{aligned} \|x^+ - x^-\| &\leqslant \|x^+ - z\| + \|z - x^-\| \\ &\leqslant tf(x^+) - tf(x^-) < \frac{1}{l}|f(x^+) - f(x^-)| \end{aligned}$$

in contradiction to the Lipschitz continuity of f.

In the following we will make use of (1) without mentioning this explicitly.

We prove now (2)–(4). It is clear that together (2)–(4) show that $C_f(t) = [C_f^+(t) \cup C] \setminus C_f^-(t)$ is convex; i.e. that for any $y_1, y_2 \in C_f(t)$ and $\xi \in [0, 1]$ we have $\xi y_1 + (1-\xi)y_2 \in C_f(t)$.

Proof of (2). We argue indirectly: Assume that $y_1, y_2 \in C_0 \setminus C_f^-(t)$ and that there exists a $\xi \in [0, 1]$ such that $y := \xi y_1 + (1-\xi)y_2 \in (x + tf(x)\mathbb{B}^\circ)$ with $x \in \partial C_0$ and $f(x) < 0$.

Let $e := x - y$, $x_1 = \eta_1 e + y_1$, $x_2 = \eta_2 e + y_2$, where $\eta_1, \eta_2 \geqslant 0$ are such that $x_1, x_2 \in \partial C_0$. Let $z := [x_1, x_2] \cap [x, y] = \xi x_1 + (1-\xi)x_2$, then

$$\begin{aligned} -\|x - y\| > tf(x) &\geqslant tf(z) - \|x - z\| \\ &\geqslant \xi tf(x_1) + (1-\xi)tf(x_2) - \|x - z\| \\ &\geqslant -\xi\|x_1 - y_1\| - (1-\xi)\|x_2 - y_2\| - \|x - z\| \\ &= -\|z - y\| - \|x - z\| = -\|x - y\|. \end{aligned}$$

The first inequality follows since $y \in (x + t\mathbb{B}^\circ)$, the second inequality follows from the Lipschitz condition, the third inequality from concavity and the fourth from the assumption that $y_1, y_2 \in C_0 \setminus C^-$. The equalities in the last two steps of the calculation follow from the definition of z. The resulting strict inequality between the first and the last term clearly is a contradiction.

Proof of (3). Assume that $y_1, y_2 \in C_f^+(t) = C_f(t) \setminus \text{int}(C_0)$. Let $x_1, x_2 \in \partial C_0$ be such that $f(x_1), f(x_2) \geqslant 0$ and $y_i \in (x_i + tf(x_i)\mathbb{B}) \setminus \text{int}(C_0)$ for $i \in \{1, 2\}$.

We distinguish two further cases:

(a) $y := \xi y_1 + (1-\xi)y_2 \notin \text{int}(C_0)$.

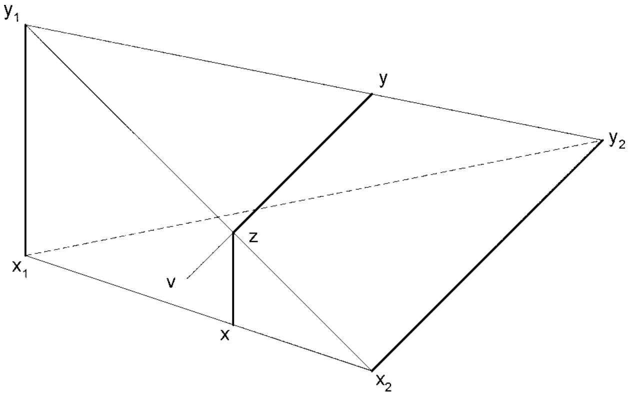

Figure 1.

Let

$$x := \xi x_1 + (1-\xi)x_2 \quad \text{and} \quad z := \partial C_0 \cap [x, y].$$

Then

$$\begin{aligned} 0 \;&\leqslant\; \|z-y\| \leqslant \xi\|x_1-y_1\| + (1-\xi)\|x_2-y_2\| - \|x-z\| \\ &\leqslant\; \xi t f(x_1) + (1-\xi) t f(x_2) - \|x-z\| \leqslant t f(x) - \|x-z\| \leqslant t f(z), \end{aligned}$$

where the second inequality follows from the relative position of the points x_1, x_2, y_1, y_2, x, y and z, the third inequality from the assumption $y_i \in (x_i + t f(x_i)\mathbb{B}) \setminus \operatorname{int}(C_0)$ for $i \in \{1, 2\}$, the penultimate inequality from the concavity of f and the last inequality from the Lipschitz condition on f. So we conclude $f(z) \geqslant 0$ and

$$y \in z + t f(z)\mathbb{B} \subseteq C_f^+(t) \subseteq [C_f^+(t) \cup C_0] \setminus C_f^-(t) = C_f(t).$$

(b) $y := \xi y_1 + (1-\xi)y_2 \in \operatorname{int}(C_0)$.

Let $y_i' \in [y_i, y] \cap \partial C_0$, then by (a) $y_i' \in C_f(t) \cap \partial C_0 \subseteq C_0 \setminus C_f^-(t)$ and, thus, we may apply the proof of (2) to y_1', y_2' (and $y := \xi' y_1' + (1 - \xi' y_2')$ for some suitable ξ').

Proof of (4). Assume that $y_1 \in C_f^+(t) \setminus C_0 = C_f(t) \setminus C_0$ and $y_2 \in C_0 \setminus C_f^-(t)$ and let x_1 be given such that $f(x_1) \geqslant 0$ and $y_1 \in (x_1 + t f(x_1)\mathbb{B})$. There are two possibilities:

(a) $y := \xi y_1 + (1-\xi)y_2 \notin \operatorname{int}(C_0)$

Let $e := y_1 - x_1$ and $x_2 := y_2 + \eta e \in \partial C_0$ with $\eta \geqslant 0$. Let $x := \xi x_1 + (1-\xi)x_2$ and $z := [x, y] \cap \partial C_0$. Then

$$\begin{aligned} t f(z) \;&\geqslant\; t f(x) - \|x-z\| \geqslant \xi t f(x_1) + (1-\xi) t f(x_2) - \|x-z\| \\ &\geqslant\; \xi\|y_1-x_1\| - (1-\xi)\|y_2-x_2\| - \|x-z\| \\ &\geqslant\; \|y-x\| - \|x-z\| \geqslant \|y-z\| \geqslant 0 \end{aligned}$$

and thus $y \in z + tf(z)\mathbb{B}$ with $f(z) \geqslant 0$; i.e. $y \in C_f^+ \subseteq C_f(t)$. Here the first inequality follows from the Lipschitz continuity of f, the second from the concavity of f, the third since we assumed $y_1 \in (x_1 + tf(x_1)\mathbb{B})$ and $y_2 \in C_0 \setminus C_f^-(t)$, which implies $\|y_1 - x_1\| \leqslant tf(x_1)$ and $-\|y_2 - x_2\| \leqslant tf(x_2)$. The fourth as well as the fifth inequality follow from the relative position of the points x_1, x_2, y_1, y_2, x, y and z.

(b) $y := \xi y_1 + (1 - \xi)y_2 \in \text{int}(C_0)$

We argue indirectly: Assume $\exists v \in \partial C_0$ with $f(v) < 0$ such that $y \in v+tf(v)\mathbb{B}$. Let $e := v-y$ and $x_2 := y_2+\eta e \in \partial C_0$, with $\eta > 0$. We set $z := [x_2, y_1]\cap\text{aff}(v, y)$ and distinguish two further cases:

I: $z = [x_2, y_1] \cap [v, y] \in C_0$ (see Figure 1).

Then

$$\begin{aligned}
&\xi tf(x_1) + (1 - \xi)tf(x_2)\\
&\quad > \xi\|x_1 - y_1\| - (1 - \xi)\|x_2 - y_2\|\\
&\quad = \xi\|x_1 - y_1\| - \|z - y\|\\
&\quad = \|(\xi x_1 + (1 - \xi)x_2) - z\| + \|z - v\| - \|v - y\|\\
&\quad \geqslant \|(\xi x_1 + (1 - \xi)x_2) - v\| + tf(v) \geqslant tf(\xi x_1 + (1 - \xi)x_2),
\end{aligned}$$

which is a contradiction to the fact that f is concave. Here the first inequality sign holds since we assumed $y_1 \in (x_1 + tf(x_1)\mathbb{B})$ and $y_2 \in C_0 \setminus C_f^-(t)$, which implies $\|y_1-x_1\| \leqslant tf(x_1)$ and $-\|y_2-x_2\| \leqslant tf(x_2)$ and the first equality sign holds, since $[z-y_1, y-y_1] = (1-\xi)[x_2-y_1, y_2-y_1]$ and thus $z-y = (1-\xi)x_2-y_2$. The second equality sign follows since $[(\xi x_1+(1-\xi)x_2)-x_2, z-x_2] = \xi[x_1-x_2, y_1-x_2]$ and thus $(\xi x_1+(1-\xi)x_2)-z = \xi(x_1-y_1)$ and since z lies on the line between v and y. The penultimate inequality holds since we assumed $y \in (v + tf(v)\mathbb{B})$, $f(v) < 0$ and thus $-\|v - y\| \geqslant tf(v)$ and the last inequality follows from the Lipschitz continuity of f.

II : $z = [x_2, y_1] \cap \text{aff}(v, y) \notin [v, y]$ and therefore $v \in [z, y]$.

This is done completely analogous to I.

For simplicity of argumentation and without loss of generality assume that $y_2 \in \text{int}(C_0)$. Let $v_2 := [x_2, y_2]\cap\text{aff}(y_1, v)$. Then we have $v_2 \in \text{int}(C_0)$. The assumptions on v imply that $v = \xi y_1 + (1 - \xi)v_2 \notin \text{int}(C_0)$. This implies together with case (a) and $f(v) < 0$ that $\forall\varepsilon > 0$ we have $v_2 \in C_f^-(\varepsilon)$. This implies further, that $v_2 \in \partial C_0$. But $v_2 \in \partial C_0$ and $v_2 \in \text{int}(C_0)$ are contradictory. (The case $y_2 \in \partial C_0$ can now be handled by a limit argument using a sequence $(y_n)_{n\in\mathbb{N}}$ of interior points of $[y_1, y_2]$ which coverges to y_2.) □

In the definition provided in Lemma 1 it is necessary to restrict the domain of a simple function $C_f(.)$ to the interval $[0, \frac{1}{l}]$. Otherwise the sets $C_f(t)$ would not be convex in general. This is shown by the following example:

EXAMPLE 13. Let

$$C_0 = \operatorname{conv}\left(\left\{\left(-\frac{1}{100}, 0\right)\right\} \cup [0, 1] \times \{-1, 1\}\right) \subset \mathbb{E}^2$$

and let

$$f(x_1, x_2) = \begin{cases} 100x_1 & \text{on } (-\infty, 0] \times \mathbb{R}, \\ 0 & \text{on } (0, \infty) \times \mathbb{R}. \end{cases}$$

Then the function $C_f\colon [0, \frac{1}{100}] \mapsto 2^{\mathbb{E}^k}$ is by Lemma 1 convex valued. But for $t \in (\frac{1}{100}, 1]$ the sets

$$\begin{aligned} C(t) &:= \Bigg[C_0 \cup \bigcup_{\{x \in \partial C_0 | f(x) \geqslant 0\}} (x + tf(x)\mathbb{B})\Bigg] \Bigg\backslash \bigcup_{\{x \in \partial C_0 | f(x) < 0\}} (x + tf(x)\mathbb{B}^\circ) \\ &= C_0 \setminus \bigcup_{\{x \in \partial C_0 | f(x) < 0\}} (x + tf(x)\mathbb{B}^\circ) \end{aligned}$$

are nonempty and compact, but not convex. (Any of them contains the points (0, 1) and (0, −1), but none of them contains the point (0, 0).)

3. The Differentiability of Simple Functions

It is rather difficult to show the weak differentiability of simple functions from the right at $t = 0$. We therefore use mixed volumes to overcome technical difficulties.

We introduce the following notations: Given a set $S \subset \mathbb{E}^k$ we denote by $\mathcal{C}(S)$ the set of convex bodies contained in S. For convex sets $U \subset \mathbb{E}^k$ or convex sets $U \subset \mathbb{E}^k \times \mathbb{R}$ we denote by $\mathcal{F}(U, l)$ respectively $\mathcal{G}(U, l)$ the space of Lipschitz continuous concave functions $f\colon U \mapsto \mathbb{R}$ respectively the space of Lipschitz continuous functions $g\colon U \mapsto \mathbb{R}$ with Lipschitz constant l endowed with the supnorm. By $O(C) := \int \mathrm{d}o_C$ we denote the surface area of C and by $\triangle(C, C') := \lambda(C \triangle C')$ we denote the symmetric difference metric on $\mathcal{C}$. Given a family of sets $\mathcal{U}$, we let $\bigcap \mathcal{U} := \bigcap_{U \in \mathcal{U}} U$ and $\bigcup \mathcal{U} := \bigcup_{U \in \mathcal{U}} U$.

If $C \in \mathcal{C}$ and μ is a measure such that $\mathrm{d}\mu = g\,\mathrm{d}\lambda$ for some continuous function g then we let

$$V'(C, f, \mu) := \int_{\partial C} f \cdot g \,\mathrm{d}o_{\partial C}$$

and call V' the volume differential of C in direction f with respect to μ. We also write $V'(C, f, g)$ instead of $V'(C, f, \mu)$. As can be easily seen from Lemma 2 the volume differential $V'(C, f, \mu)$ describes the infinitesimal change of the volume $V(C_f(t))$ of the values of the simple function $C_f(.)$ at $t = 0$. Continuity properties of $V'(.,.,.)$ are proved in the Appendix.

We call a convex body which is a polyhedron a polyhedral body. We say that a simple set valued function $P_f(.)$ is polyhedral if $P_f(0) = P_0$ is a polyhedral body. We prove the differentiability lemma (Lemma 2) for polyhedral simple functions first and then extend it to arbitrary simple functions. To show the differentiability of a polyhedral simple function $P_f(.)$ at $t = 0$ we need the Propositions 1–3 concerning the growth of polyhedra and the deformation of faces of polyhedra.

PROPOSITION 1. *Let $P_0 \subset \mathbb{E}^k$ be a k-dimensional convex polytope with $r\mathbb{B} \subset P_0 \subset R\mathbb{B}$. Let F_i denote its $(k-1)$-dimensional facets, let e_i be the outward unit normal vector onto F_i, and let for $t \geqslant 0$*

$$F_i^t := \{x + \xi e_i \mid x \in F_i \wedge \xi \in [0, t]\}$$

and

$$F_i^{-t} := \{x - \xi e_i \mid x \in F_i \wedge \xi \in [0, t]\}.$$

Then $\forall t \in [0, r^2/2R)$ we have

$$V\left(\bigcup_{i \neq j}(F_i^{-t} \cap F_j^{-t})\right) \leqslant t^2 L_2$$

and

$$V\left((P_0 + t\mathbb{B}) \setminus \left(P_0 \cup \bigcup_i F_i^t\right)\right) \leqslant t^2 L_2,$$

where L_2 depends on (k, r, R) only is given by $L_2(k, r, R) = 3^k V((R + R/r)\mathbb{B})$.

Proof. We have

$$\begin{aligned}
(1)\quad & V\left(\left((P_0 \ominus t\mathbb{B}) + t\frac{R}{r}\mathbb{B} + t\mathbb{B}\right) \Big\backslash \left((P_0 \ominus t\mathbb{B}) + t\frac{R}{r}\mathbb{B}\right)\right) \\
& \geqslant V((P_0 + t\mathbb{B}) \setminus P_0) \geqslant \sum_{i=1}^{k} V(F_i^t) = \sum_{i=1}^{k} V(F_i^{-t}) \\
& = V(P_0 \setminus (P_0 \ominus t\mathbb{B})) + V\left(\bigcup_{i \neq j}(F_i^{-t} \cap F_j^{-t})\right) \\
(2)\quad & \geqslant V(((P_0 \ominus t\mathbb{B}) + t\mathbb{B}) \setminus (P_0 \ominus t\mathbb{B})) + V\left(\bigcup_{i \neq j}(F_i^{-t} \cap F_j^{-t})\right),
\end{aligned}$$

where the first inequality follows from Proposition A.3 of the Appendix and the fact that $P_0 \subseteq (P_0 \ominus t\mathbb{B}) + t(R/r)\mathbb{B}$, the second since the sets F_i^t, F_j^t have for $i \neq j$ not more than one point in common, $P_0 + t\mathbb{B} \setminus \mathrm{int}(P_0) \supset \bigcup_i F_i^t$ and $V(\partial P_0) = 0$. The second equality follows since $\bigcup F_i^{-t} = P_0 \setminus \mathrm{int}(P_0 \ominus t\mathbb{B})$ and $V(\partial(P_0 \ominus t\mathbb{B}) = 0$.

Finally the third and last inequality follows since $((P_0 \ominus t\mathbb{B}) + t\mathbb{B}) \setminus (P_0 \ominus t\mathbb{B}) \subseteq P_0 \setminus (P_0 \ominus t\mathbb{B})$. Thus we get from the inequality between the expressions (1) and (2)

$$(3)\quad V\left(\bigcup_{i\neq j}(F_i^{-t} \cap F_j^{-t})\right) \leqslant \begin{cases} V\left(\left((P_0 \ominus t\mathbb{B}) + t\frac{R}{r}\mathbb{B} + t\mathbb{B}\right) \Big\backslash \left((P_0 \ominus t\mathbb{B}) + t\frac{R}{r}\mathbb{B}\right)\right), \\ -V(((P_0 \ominus t\mathbb{B}) + t\mathbb{B}) \setminus (P_0 \ominus t\mathbb{B})). \end{cases}$$

Further, we have

$$(4)\quad V\left((P_0 \ominus t\mathbb{B}) + t\frac{R}{r}\mathbb{B} + t\mathbb{B}\right) = \sum_{i_1,\ldots,i_k \in \{1,2,3\}} v(Q_{i_1}, \ldots, Q_{i_k}),$$

where

$$Q_1 := P_0 \ominus t\mathbb{B}, \quad Q_2 := t\frac{R}{r}\mathbb{B} \quad \text{and} \quad Q_3 := t\mathbb{B}$$

and $v(Q_{i_1}, \ldots, Q_{i_k})$ denotes the mixed volume of $Q_{i_1}, \ldots, Q_{i_k}$. This is obtained directly from properties (i)–(iv) in Remark A.1 of the Appendix of mixed volumes (see also [11, 17]).

From these properties of mixed volumes we also get

$$(5)\quad V\left((P_0 \ominus t\mathbb{B}) + t\frac{R}{r}\mathbb{B}\right) = \sum_{i_1,\ldots,i_k \in \{1,2\}} v(R_{i_1}, \ldots, R_{i_k}),$$

where $R_1 := P_0 \ominus t\mathbb{B}$ and $R_2 := t(R/r)\mathbb{B}$,

$$(6)\quad V((P_0 \ominus t\mathbb{B}) + t\mathbb{B}) = \sum_{i_1,\ldots,i_k \in \{1,2\}} v(S_{i_1}, \ldots, S_{i_k}),$$

where $S_1 := P_0 \ominus t\mathbb{B}$ and $S_2 := t\mathbb{B}$ and finally

$$V(P_0 \ominus t\mathbb{B}) = v(P_0 \ominus t\mathbb{B}, \ldots, P_0 \ominus t\mathbb{B}).$$

Thus we get

$$\begin{aligned}(7)\quad & V\left(\left((P_0 \ominus t\mathbb{B}) + t\frac{R}{r}\mathbb{B} + t\mathbb{B}\right) \Big\backslash \left((P_0 \ominus t\mathbb{B}) + t\frac{R}{r}\mathbb{B}\right)\right) - \\ & - V(((P_0 \ominus t\mathbb{B}) + t\mathbb{B}) \setminus (P_0 \ominus t\mathbb{B})) \\ & = \left(\sum_{i_1,\ldots,i_k \in \{1,2,3\}} v(Q_{i_1}, \ldots, Q_{i_k}) - \sum_{i_1,\ldots,i_k \in \{1,2\}} v(R_{i_1}, \ldots, R_{i_k})\right) - \\ & - \left(\sum_{i_1,\ldots,i_k \in \{1,2\}} v(S_{i_1}, \ldots, S_{i_k}) - v(P_0 \ominus t\mathbb{B}, \ldots, P_0 \ominus t\mathbb{B})\right)\end{aligned}$$

and easy calculation shows that

(8) the last expression equals $\sum_{(i_1,\ldots,i_k)\in I} v(Q_{i_1}, \ldots, Q_{i_k})$,

where a tuple $(i_1, \ldots, i_k)$ lies in I if and only if (at least) one of the numbers $i_1, \ldots, i_k$ equals 2 and (at least) one equals 3; i.e. if and only if one of the sets $Q_{i_1}, \ldots, Q_{i_k}$ equals $t(R/r)\mathbb{B}$ and another one equals $t\mathbb{B}$.

Thus by the rules for manipulation of mixed volumes (see Remark A.1 of the Appendix) and the fact that $P_0 \ominus t\mathbb{B} \subseteq R\mathbb{B}$ we get

$$\text{(9)} \qquad \sum_{(i_1,\ldots,i_k)\in I} v(Q_{i_1}, \ldots, Q_{i_k}) \leqslant t^2 3^k V\left(\left(R + \frac{R}{r}\right)\mathbb{B}\right).$$

So finally we get from (3), (7), (8) and (9) that

$$V\left(\bigcup_{i\neq j}(F_i^{-t} \cap F_j^{-t})\right) \leqslant t^2 3^k V\left(\left(R + \frac{R}{r}\right)\mathbb{B}\right) = t^2 L_2(k, r, R).$$

Similarly, the inequality

$$V\left((P_0 + t\mathbb{B}) \setminus \left(P_0 \cup \bigcup_i F_i^t\right)\right) \leqslant t^2 L_2(k, r, R)$$

can be proved and thus Proposition 1 is proved. □

The next proposition concerns the faces F_i of a polyhedral body P_0. In the proposition and its proof we define sets $F_i(t)$, $G_i(t)$ and $H_i(t)$ and functions f_i^t, g_i^t and h_i^t which model the growth of a set valued function $P_f(t)$ along each face F_i up to order 1. The sets $F_i(t)$ are defined by adding half balls to F_i, the sets $H_i(t)$ are defined by adding half squares to F_i and the sets $G_i(t)$ are defined by adding line segments to F_i. The functions f_i^t, g_i^t and h_i^t only take on the values -1, 0 and 1. They tell us which regions of the sets F_i, G_i and H_i should be counted positive (added to P_0) or negative (subtracted from P_0).

PROPOSITION 2. *Let $F_i \subset \mathbb{E}^k$ be a $(k-1)$-dimensional face of a polyhedral body P_0 with nonempty interior and let $e_i \in \mathbb{E}^k$ be the outward unit normal vector onto F_i. Let $f_i\colon F_i \mapsto \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant l (i.e. the restriction of a Lipschitz continuous convex function $f\colon P_0 \mapsto \mathbb{R}$ to F_i). For $t \in [0, \frac{1}{l}]$ let*

$$G_i(t) := \{x + \xi e_i \mid x \in F_i \mid \operatorname{sgn}(\xi) = \operatorname{sgn}(f_i(x)) \text{ and } |\xi| \leqslant t|f_i(x)|\}$$

and

$$F_i(t) := \{x + \xi e_i \mid x \in F_i,\ \exists y \in F_i \mid (x-y)^2 + \xi^2 \leqslant (tf_i(y))^2 \ \wedge\ \operatorname{sgn}(\xi) = \operatorname{sgn} f_i(y)\}.$$

Define

$$f_i^t\colon \mathbb{E}^k \mapsto \mathbb{R} \quad \textit{by } f_i^t(x+\xi e_i) := \begin{cases} \operatorname{sgn}(\xi) & \textit{for } x+\xi e_i \in F_i(t), \\ f_i^t = 0 & \textit{otherwise} \end{cases}$$

and

$$g_i^t\colon \mathbb{E}^k \mapsto \mathbb{R} \quad \textit{by } g_i^t(x+\xi e_i) := \begin{cases} \operatorname{sgn}(\xi) & \textit{for } x+\xi e_i \in G_i(t), \\ g_i^t = 0 & \textit{otherwise}. \end{cases}$$

Then for any continuous function φ with compact support we get

$$\int |\varphi(f_i^t - g_i^t)|\,\mathrm{d}\lambda \leqslant t^2 l \|f_i\| \|\varphi\| V(F_i).$$

Proof. Let

$$\begin{aligned} H_i(t) \;:=\; & \{x+\xi e_i \mid x \in F_i;\, \exists y \in F_i \mid \max(\|x-y\|, \xi) \\ & \leqslant t|f_i(y)| \wedge \operatorname{sgn}(\xi) = \operatorname{sgn}(f_i(y))\} \end{aligned}$$

and define

$$h_i^t\colon \mathbb{E}^k \mapsto \mathbb{R} \quad \text{by } h_i^t(x+\xi e_i) = \begin{cases} \operatorname{sgn}(\xi) & \text{for } x+\xi e_i \in H_i(t), \\ h_i^t = 0 & \text{otherwise}. \end{cases}$$

Then $|g_i^t - f_i^t| \leqslant |g_i^t - h_i^t|$ and thus (we denote by λ_{F_i} the area measure of F_i)

$$\begin{aligned} \int |\varphi(g_i^t - f_i^t)|\,\mathrm{d}\lambda \;&\leqslant\; \int |\varphi(g_i^t - h_i^t)|\,\mathrm{d}\lambda \leqslant \|\varphi\| \triangle(G_i(t), H_i(t)) \\ &\leqslant\; \|\varphi\| \int t\Big(\sup_{\{x \mid \|y-x\| \leqslant t\cdot|f_i(x)|\}} |f_i(x)| - |f_i(y)|\Big)\,\mathrm{d}\lambda_{F_i}(y) \\ &\leqslant\; \|\varphi\| \int t\Big(\sup_{\{x \mid \|y-x\| \leqslant t\cdot\|f_i\|\}} |f_i(x)| - |f_i(y)|\Big)\,\mathrm{d}\lambda_{F_i}(y) \\ &\leqslant\; \|\varphi\| \int t^2 l \|f\|\,\mathrm{d}\lambda_{F_i} \leqslant t^2 l \|\varphi\| \cdot \|f_i\| V(F_i). \end{aligned}$$

For the third inequality see Figure 2 and note that by the Lipschitz continuity of f we have for $x \in F_i$

$$\xi, \eta \neq 0, \quad (x,\xi) \in G_i(t) \wedge (x,\eta) \in H_i(t) \Rightarrow \operatorname{sgn}(\xi) = \operatorname{sgn}(\eta).$$

The penultimate inequality also follows from the Lipschitz continuity of f. □

In the following proposition we make use of the notations introduced in the Propositions 1 and 2.

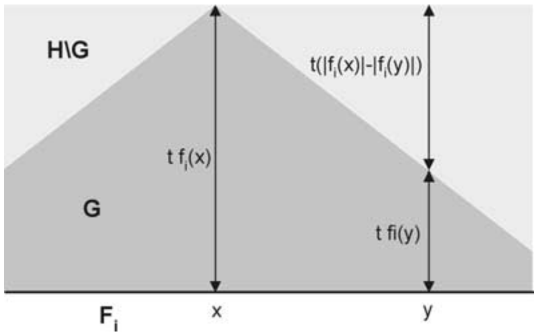

Figure 2.

PROPOSITION 3. *Let P_0 be a polyhedral body in $\mathbb{E}^k$ with faces F_i, let $r\mathbb{B} \subset P_0 \subset R\mathbb{B}$, $f \in \mathcal{F}(P_0, l)$, then for t sufficiently small (depending only on l, k, r, R) we get*

$$\int \left\| \mathbb{1}_{P_f(t)} - \mathbb{1}_{P_0} - \sum f_i^t \right\| \mathrm{d}\lambda \leqslant 2t^2 L_2 \|f\|^2.$$

Proof. Let

$$T := t\|f\|, \quad J^T := P_0 + T\mathbb{B} \setminus \left(P_0 \cup \bigcup_i F_i^T \right)$$

and

$$J^{-T} := \bigcup_{i \neq j} (F_i^T \cap F_j^{-T}).$$

With these definitions we get

$$\left\| \mathbb{1}_{P_f(t)} - \mathbb{1}_{P_0} - \sum f_i^t \right\| \leqslant \mathbb{1}_{J^T} + \mathbb{1}_{J^{-T}} + \mathbb{1}_{\partial P_f(t)} + \mathbb{1}_{\partial P_0} + \mathbb{1}_{\partial\, \mathrm{supp}(f_i^t)}$$

and thus we get the assertion from Proposition 1 and the fact that the integral over the last 3 terms of our inequality vanishes. □

Now we are in the position to prove the fundamental differentiability lemma for simple functions using Propositions 2 and 3. (The intuitive meaning of Propositions 2 and 3 together is that the growth of a polyhedral simple function $P_f(.)$ can be equivalently described up to order 1 by the addition and subtraction of certain balls or certain line segments perpendicular to the faces of the polyhedral body $P_f(0)$.)

LEMMA 2. *Let $C \in (\mathcal{C}, d)$ and let U be a bounded convex neighborhood of C, let $B_{\mathcal{F}} \subset \mathcal{F}(U, l)$ and $B_{\mathcal{G}} \subset \mathcal{G}(U, l)$ be bounded sets. Then there exists a neighborhood $\mathcal{U}(C) \subset (\mathcal{C}, d)$ such that $\forall (C_0, f, g) \in \mathcal{U}(C) \times B_{\mathcal{F}} \times B_{\mathcal{G}}$ the functions C_f*

defined in Lemma 1 *are weak equidifferentiable at* 0 *with respect to the measures* μ *with* $\mathrm{d}\mu = g\,\mathrm{d}\lambda$. *The weak derivative* $\dot{C}_f^\mu(0)$ *is the measure on* ∂C *such that*

$$\frac{\mathrm{d}\dot{C}_f^\mu(0)}{\mathrm{d}o_{C_0}} = f \cdot g|_{\partial C_0} \quad \textit{almost everywhere } o_{\partial C_0}.$$

To be more precise: For any continuous function $\psi\colon \mathbb{E}^k \mapsto \mathbb{R}$ *with compact support there exists a continuous function* $\varepsilon_\psi\colon\ [0,\infty) \mapsto [0,\infty)$ *with fixed point* 0, *which is independent of* (C_0, f, g) *such that*

$$\int \psi \left[\frac{\mathrm{d}\mu|_{C_f(t)} - \mathrm{d}\mu|_{C_0}}{t} - \mathrm{d}\dot{C}_f^\mu(0) \right] < \varepsilon_\psi(t).$$

Proof. Since U is bounded we have $U \subset R\mathbb{B}$ for some $R > 0$. Let us choose $\mathcal{U}(C)$ such that $\bigcap \mathcal{U}(C)$ contains a ball B of radius r and $\bigcup \mathcal{U}(C) \subset U$. We first show the assertion of the theorem for the set of polyhedral bodies $\mathcal{P}$ contained in $\mathcal{U}(C)$. So let $(P_0, f, g) \in \mathcal{U}(C) \times B(\mathcal{F}) \times B(\mathcal{G})$ and ψ be given, let $\{F_i \mid i\}$ be the set of faces of the polyhedral body P_0 and set $\varphi = \psi \cdot g$. Then

$$\begin{aligned}
&\int \psi \left[\frac{\mathrm{d}\mu|_{P_f(t)} - \mathrm{d}\mu|_{P_0}}{t} - \mathrm{d}\dot{P}_f^\mu(0) \right] \\
&\quad = \int \varphi \frac{\mathbb{1}_{P_f(t)} - \mathbb{1}_{P_0}}{t} \mathrm{d}\lambda - \int \varphi \cdot f \,\mathrm{d}o_{P_0} \\
&\quad \text{(by Proposition 3)} \\
&\quad \leqslant \frac{\|\varphi\| \cdot 2t^2 \|f\|^2 L_2 + \sum_i \int \varphi \cdot f_i^t \,\mathrm{d}\lambda}{t} - \int \varphi \cdot f \,\mathrm{d}o_{P_0} \\
&\quad \leqslant 2t\|\varphi\| \cdot \|f\|^2 L_2 + \sum_i \int \frac{\|\varphi\| \cdot |f_i^t - g_i^t|}{t} \mathrm{d}\lambda + \\
&\qquad + \sum_i \int \frac{\varphi \cdot g_i^t}{t} \mathrm{d}\lambda - \int \varphi \cdot f \,\mathrm{d}o_{P_0}.
\end{aligned}$$

By Proposition 2 and since the last two terms are equal up to the sign we see, that the last expression is smaller than

$$\begin{aligned}
&2t\|\varphi\| \cdot \|f\|^2 L_2(k, r, R) + \sum_i tl\|\varphi\| \cdot \|f_i\| \cdot V(F_i) \\
&\quad \leqslant 2t\|\psi\| \cdot \sup_{g \in B_{\mathcal{G}}} \|g\| \cdot \Big(\sup_{f \in B_{\mathcal{F}}} (\|f\|^2 \cdot L_2 + l \cdot \|f\| \cdot \mathrm{O}(P_0)) \Big).
\end{aligned}$$

Setting

$$\varepsilon_\psi(t) = 2t\|\psi\| \cdot \sup_{g \in B_{\mathcal{G}}} \|g\| \cdot \Big(\sup_{f \in B_{\mathcal{F}}} (\|f\|^2 \cdot L_2 + l \cdot \|f\| \cdot \mathrm{O}(P_0)) \Big)$$

this proves the lemma for polyhedral bodies. □

We further note that for a given t the set $C_f(t)$ depends continuously on C_0 (with respect to the Hausdorff metric) and thus (by a use of Lemma A.2 of the Appendix) that

$$\int \psi\left[\frac{\mathrm{d}\mu|_{C_f(t)} - \mathrm{d}\mu|_{C_0}}{t} - \mathrm{d}\dot{C}^{\mu}_f(0)\right] = \int \varphi \frac{\mathbb{1}_{C_f(t)} - \mathbb{1}_{C_0}}{t}\,\mathrm{d}\lambda - \int_{\partial C_0} \varphi \cdot f\,\mathrm{d}o|_{\partial C_0}$$

is continuous in C_0. This together with the fact that the space of polyhedral bodies is dense in the space of convex bodies shows the validity of the theorem. □

4. Approximation of Solutions

Finally we have to approximate the solutions by "polygons". This is complicated by the fact that the function $f - \kappa$ which occurs in Theorem 1 is time dependent even if f is not. Therefore we decided to prove Theorem 1 in the general case of a time dependent function f, as stated in the Introduction. We also present a more general Theorem (Theorem 1′) where not only f, but also the measure μ depends on time and the volume of the sets $C(t)$ of our set valued solution function equals the value of a previously given function $\alpha(t)$ at any time t. (We state Theorem 1′ without proof. It can be proved in a way completely analogous to the proof of Theorem 1.)

DEFINITION. Given $C_0 \in \mathcal{C}$ a sufficiently large open set $U \supset C_0$ concave functions $f_1, \ldots, f_n \in \mathcal{F}(U, l)$ and numbers $s_1, \ldots, s_n \in [0, \frac{1}{l})$ we denote by $C_{f_1}(s_1)_{f_2}(s_2)\ldots_{f_n}(s_n)$ the convex set which is obtained from C_0 by first changing C_0 to $C_1 := C_{f_1}(s_1)$ then C_1 to $C_2 := C_{f_2}(s_2)$ and so on. To be more precise we define $C_{f_1}(s_1)_{f_2}(s_2)\ldots_{f_n}(s_n)$ by recursion on $n \in \mathbb{N}$ as follows: For $n = 1$ let $C_{f_1}(s_1)$ be the function $C_f(.)$ of Lemma 1 with f replaced by f_1 at the point s_1. Now suppose that $C_{f_1}(s_1)_{f_2}(s_2)\ldots f_{n-1}(s_{n-1})$ has already been defined. Let

$$C_{f_1}(s_1)_{f_2}(s_2)\ldots_{f_n}(s_n) = (C_{f_1}(s_1)_{f_2}(s_2)\ldots_{f_{n-1}}(s_{n-1}) \cup C^n_+) \setminus C^n_-$$

with

$$C^n_+ := \bigcup_{\{x\in\partial C_{f_1}(s_1)_{f_2}(s_2)\ldots_{f_{n-1}}(s_{n-1}) | f_n(x)\geqslant 0\}} (x + s_n f_n(x)\mathbb{B}),$$

$$C^n_- := \bigcup_{\{x\in\partial C_{f_1}(s_1)_{f_2}(s_2)\ldots_{f_{n-1}}(s_{n-1}) | f_n(x)< 0\}} (x + s_n f_n(x)\mathbb{B}^\circ).$$

In the following $\mathbb{B}_r(x)$ denotes the unit ball of radius r with center x.

LEMMA 3. *Let $l \geqslant 1$. Let $C_0 \in (\mathcal{C}, d)$, let U be a bounded neighborhood of C_0 and $f \in \mathcal{G}([0, 1/l] \times U, l)$ be such that $\forall s \in [0, 1/l]$ the function $f(s, .)$ is* *concave. Let further* $s \in [0, 1/l)$ *and assume that*

(i) $r\mathbb{B} \subset C_{f(0,.)}(s) \subset U \subset R\mathbb{B}$,
$r\mathbb{B} \subset C_{f(0,.)}(s)_{f(0,.)-sl(\|f\|+1)}(t) \subset C_{f(0,.)}(s)_{f(0,.)+sl(\|f\|+1)}(t) \subset U \subset R\mathbb{B}$

then for all $t \in [0, (1/l) - s)$ *we have*

$$d(C_{f(0,.)}(s)_{f(s,.)}(t), C_{f(0,.)}(s+t)) \leqslant ts \cdot 2l(\|f\|+1)\frac{R}{r}\mathbb{B}.$$

Further, if

(ii) *for* $s, t > 0,\ m, \tilde{m} \in \mathbb{N},\ ms < t < 1/l$ *and for all* $\tilde{m} \leqslant m$,

$$r\mathbb{B} \subset C_{f(0,.)}(s)_{f(0,.)-sl(\|f\|+1)}(s)\ldots_{f(0,.)-(\tilde{m}-1)sl(\|f\|+1)}(s) \subset C_{f(0,.)}(s)_{f(0,.)+sl(\|f\|+1)}(s)\ldots_{f(0,.)+(\tilde{m}-1)sl(\|f\|+1)}(s) \subset U \subset R\mathbb{B}$$

holds, then

$$d(C_{f(0,.)}(t), C_{f(0,.)}(s)_{f(s,.)}(s)\ldots_{f(ms,.)}(t-ms)) \leqslant t^2 \cdot 2l(\|f\|+1)\frac{R}{r}.$$

Before we prove the lemma we give a very simple example which helps to clarify the situation.

EXAMPLE 14. Let $f\colon \mathbb{R} \times \mathbb{E}^k \mapsto \mathbb{R}$ be given by $f(t,x) = c + t$ for some fixed $c \in \mathbb{R}$ and let $C_0 = \mathbb{B} \subset \mathbb{E}^k$. Then

$$C_{f(0,.)}(s+t) = 1 + c(s+t)\mathbb{B} \quad \text{and} \quad C_{f(0,.)}(s)_{f(s,.)}(t) = (1 + cs + (c+s)t)\mathbb{B}$$

so that

$$d(C_{f(0,.)}(s+t), C_{f(0,.)}(s)_{f(s,.)}(t)) \leqslant s \cdot t.$$

From this it is easy to derive

$$\begin{aligned} &d(C_{f(0,.)}(s)_{f(s,.)}(s)\ldots_{f(\tilde{m}s,.)}(t-\tilde{m}s),\\ &\qquad C_{f(0,.)}(s)_{f(s,.)}(s)\ldots_{f((\tilde{m}-1)s,.)}(t-(\tilde{m}-1)s))\\ &\quad \leqslant (t-\tilde{m}s)s \end{aligned}$$

for any $\tilde{m}$ with $\tilde{m}s \leqslant t$. Adding these inequalities and using the triangle inequality for the Hausdorff metric d we obtain

$$d(C_{f(0,.)}(t), C_{f(0,.)}(s)_{f(s,.)}(s)\ldots_{f(ms,.)}(t-ms)) \leqslant \sum_{\tilde{m}=1}^{m}(t-\tilde{m}s)s \leqslant t^2.$$

Proof of Lemma 3. To prove $d(C_{f(0,.)}(s)_{f(s,.)}(t), C_{f(0,.)}(s+t)) \leqslant ts \cdot 2l \times (\|f\|+1)R/r$ it is clearly sufficient to show that

$$\begin{aligned}(1)\quad & C_{f(0,.)}(s)_{f(0,.)}(t)_{-sl(\|f\|+1)}(t)\\ &= C_{f(0,.)}(s)_{f(0,.)-sl(\|f\|+1)}(t)\\ &\subseteq C_{f(0,.)}(s+t) \subseteq C_{f(0,.)}(s)_{f(0,.)+sl(\|f\|+1)}(t)\\ &= C_{f(0,.)}(s)_{f(0,.)}(t)_{+sl(\|f\|+1)}(t)\\ &\subseteq C_{f(0,.)}(s)_{f(0,.)}(t)_{-sl(\|f\|+1)}(t) + 2st \cdot l\frac{R}{r}\mathbb{B}\end{aligned}$$

and

$$\begin{aligned}(2)\quad & C_{f(0,.)}(s)_{f(0,.)}(t)_{-sl(\|f\|+1)}(t)\\ &= C_{f(0,.)}(s)_{f(0,.)-sl(\|f\|+1)}(t)\\ &\subseteq C_{f(0,.)}(s)_{f(s,.)}(t) \subseteq C_{f(0,.)}(s)_{f(0,.)+sl(\|f\|+1)}(t)\\ &= C_{f(0,.)}(s)_{f(0,.)}(t)_{+sl(\|f\|+1)}(t)\\ &\subseteq C_{f(0,.)}(s)_{f(0,.)}(t)_{-sl(\|f\|+1)}(t) + 2st \cdot l\frac{R}{r}\mathbb{B}.\end{aligned}$$

Since by hypothesis (i)

$$r\mathbb{B} \subset C_{f(0,s)}(s)_{f(0,.)-sl(\|f\|+1)}(t) \subset R\mathbb{B}$$

and since we have by Lipschitz continuity of f

$$f(0,.) - sl(\|f\|+1) \leqslant f(s,.) \leqslant f(0,.) + sl(\|f\|+1)$$

all the inclusions in (2) are obvious. In (1) all the inclusions except

$$(3)\quad C_{f(0,.)}(s)_{f(0,.)-sl(\|f\|+1)}(t) \subseteq C_{f(0,.)}(s+t)$$

and

$$(4)\quad C_{f(0,.)}(s+t) \subseteq C_{f(0,.)}(s)_{f(0,.)+sl(\|f\|+1)}(t)$$

are obvious, so that it remains to show (3) and (4).

We show (3) first.

Let $x \in \partial(C_{f(0,.)}(s+t))$. We have to show that

$$(5)\quad x \notin \text{int}(C_{f(0,.)}(s)_{f(0,.)-sl(\|f\|+1)}(t))$$

and distinguish two cases.

First case: $x \in \text{int}(C_0)$.
Second case: $x \notin \text{int}(C_0)$.

In the *first case* there exists $y \in \partial C_0$ such that $\|x-y\| = -(s+t)f(0,y)$. Let $w := \partial\mathbb{B}_{sf(0,y)}(y) \cap [x,y]$ and $z := \partial(C_{f(0,.)}(s)) \cap [x,y]$. (See Figure 3.)

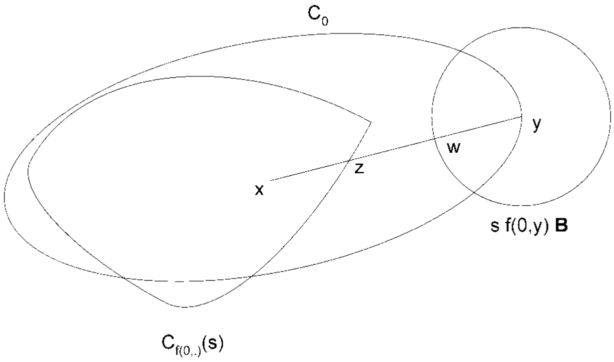

Figure 3.

Then we have

$$-\|w-y\| = sf(0,y) \quad \text{and} \quad \|x-w\| = -tf(0,y) \tag{6}$$

and thus

$$\begin{aligned}
\|x-z\| &= \|x-w\| - \|w-z\| = -tf(0,y) - \|w-z\| \\
&\leqslant -t(f(0,w) - |f(0,w) - f(0,y)|) - \|w-z\| \\
&\leqslant -t(f(0,w) - l\|w-y\|) - \|w-z\| \\
&= -t(f(0,w) + slf(0,y)) - \|w-z\| \\
&\leqslant -t(f(0,z) - |f(0,z) - f(0,w)| - sl\|f\|) - \|w-z\| \\
&\leqslant -t(f(0,z) - l\|w-z\| - sl\|f\|) - \|w-z\| \\
&= -t(f(0,z) - sl\|f\|) - (1-tl)\|w-z\| \\
&\leqslant -t(f(0,z) - sl\|f\|) \leqslant -t(f(0,z) - sl(\|f\|+1)).
\end{aligned}$$

Here the second and fourth inequality follow from Lipschitz continuity of f, the second and third equality follow from (6) and the penultimate inequality follows since $tl \leqslant 1$. So

$$\|x-z\| \leqslant -t(f(0,z) - sl(\|f\|+1))$$

and thus

$$x \notin \operatorname{int}(C_{f(0,.)}(s)_{f(0,.)-sl(\|f\|+1)}(t)),$$

i.e. (5) has been proved.

In the *second case* we proceed indirectly. We assume that

$$x \in \operatorname{int}(C_{f(0,.)}(s)_{f(0,.)-sl(\|f\|+1)}(t)).$$

Since $x \notin \operatorname{int}(C_0)$ there exists a $z \in \partial C_0$ and a $w \in \partial\mathbb{B}_{sf(0,z)}(z)$ such that

$$x \in \mathbb{B}^{\circ}_{t(f(0,w)-sl(\|f\|+1))}(w).$$

Since by Lipschitz continuity $|f(0,z) - f(0,w)| \leqslant l\|z-w\| = ls$ holds, we get further

$$x \in \mathbb{B}^\circ_{t(f(0,w)-sl(\|f\|+1))}(w) \subseteq \mathbb{B}^\circ_{tf(0,z)}(w).$$

But by

$$w \in \partial\mathbb{B}_{sf(0,z)}(z) \ \text{ and } \ x \in \mathbb{B}^\circ_{tf(0,z)}(w)$$

we get

$$x \in \mathbb{B}^\circ_{(s+t)f(0,z)}(z)$$

which contradicts $x \in \partial C_{f(0,.)}(s+t)$.

Thus (3) is proved.

To show (4) we again let $x \in \partial(C_{f(0,.)}(s+t))$ and distinguish again the cases $x \in \text{int}(C_0)$ and $x \notin \text{int}(C_0)$.

In the first case we proceed indirectly analogous to the second case in the proof of (3). We assume that

$$x \notin C_{f(0,.)}(s)_{f(0,.)+sl(\|f\|+1)}(t)$$

and note that there exists $w \in \partial\mathbb{B}_{sf(0,z)}(z)$ such that $x \in \mathbb{B}^\circ_{t(f(0,w)+sl(\|f\|+1))}(w)$. As in the second case of the proof of (3) we get a contradiction by showing that $x \in \mathbb{B}^\circ_{(s+t)f(0,z)}(z)$.

The second case of the proof of (4) is analogous to the first case of the proof of (3). Since $x \notin \text{int}(C_0)$ there exists a $y \in \partial C_0$ with $\|x-y\| = (s+t)f(0,y)$. We again let $w := \partial\mathbb{B}_{sf(0,y)}(y) \cap [x,y]$ and $z := \partial(C_{f(0,.)}(s)) \cap [x,y]$. Then by a calculation analogous to the calculation which proves the first case of (3) we get

$$\|x-z\| \leqslant t(f(0,z) + sl(\|f\|+1))$$

and thus that $x \in C_{f(0,.)}(s)_{f(0,.)+sl(\|f\|+1)}$, which completes the proof of (4).

To show that

$$d(C_{f(0,.)}(t), C_{f(0,.)}(s)_{f(s,.)}(s)\ldots_{f(ms,.)}(t-ms)) \leqslant t^2 \cdot 2l\frac{R}{r}(\|f\|+1)$$

we note that (by what we have just proved) for all $\tilde{m} \leqslant m$

$$\begin{aligned} &d(C_{f(0,.)}(s)_{f(s,.)}(s)\ldots_{f(\tilde{m}s,.)}(t-\tilde{m}s), \\ &\qquad C_{f(0,.)}(s)_{f(s,.)}(s)\ldots_{f((\tilde{m}-1)s,.)}(t-(\tilde{m}-1)s)) \\ &\quad \leqslant (t-\tilde{m}s)s \cdot 2l\frac{R}{r}(\|f\|+1) \end{aligned}$$

and thus summing up this equations we obtain (analogous to Example 14) that

$$\begin{aligned} &d(C_{f(0,.)}(t), C_{f(0,.)}(s)_{f(s,.)}(s)\ldots_{f(ms,.)}(t-ms)) \\ &\quad \leqslant \sum_{\tilde{m}=1}^{m}(t-\tilde{m}s)s \cdot 2l\frac{R}{r}(\|f\|+1) \leqslant t^2 \cdot 2l\frac{R}{r}(\|f\|+1). \qquad \square \end{aligned}$$

THEOREM 1. *Let* $C_0 \in (\mathcal{C}, d)$ *and let* U *be a compact convex neighborhood of* C_0*. Let* $f \in \mathcal{G}([0, \hat{\rho}] \times U, l)$ *be such that for any* t *the function* $f(t, .)$ *is concave and let* μ *be a measure on* U *with strictly positive Lipschitz continuous density* $g \in G(U, l)$ *with respect to Lebesgue measure* λ *on* U*. Then there exists* $\rho \in (0, \hat{\rho}]$ *and a set-valued mapping* $C\colon [0, \rho) \mapsto \mathcal{C}$ *with convex compact values and a function* $\kappa\colon [0, \rho) \mapsto \mathbb{R}$ *such that* $C(0) = C_0$*,* $\mu(C(t)) = \mu(C_0)$ *and*

$$\frac{\mathrm{d}(\dot{C}^{\mu}(t))}{\mathrm{d}(o_{C(t)})} = (f(t, .) - \kappa(t))g(.),$$

where $\kappa(t)$ *fulfills*

$$\int_{\partial C(t)} g(.)\kappa(t)\,\mathrm{d}(o) = \int_{\partial C(t)} f(t, .) \cdot g(.)\,\mathrm{d}o_{\partial C(t)},$$

or equivalently (*using the notation of volume differential*)

$$\kappa(t) = \frac{V'(C(t), f(t, .), \mu)}{V'(C(t), 1, \mu)}.$$

Remark. As already mentioned in the Introduction the function κ is necessary for the volume correction; i.e. κ forces the set valued function C to satisfy $\mu(C(t)) = \mu(C_0)$ for all $t \in [0, \rho)$. How this can be used for constraint stochastic optimization has already been shown in the Examples 6 and 7 which together show how an optimal statistical test can be obtained as the limit $\lim_{t\to\infty} C(t)$ of a set valued function $C\colon [0, \infty) \mapsto \mathcal{C}(\mathbb{E}^k)$. An example in which $\kappa(.)$ is of a very simple form which can be easily parameterized is the following:

Let $C_0 := r\mathbb{B}$, let $f(t, x_1, \ldots, x_k) := x_1$ and let $\mu := \lambda$. Then (see also Example 2) a solution $C\colon [0, \infty) \mapsto \mathcal{C}(\mathbb{E}^k)$ with $C(0) = C_0$ and $\mu(C(t)) = \mu(C_0)$ of

$$\frac{\mathrm{d}\dot{C}^{\mu}(t)}{\mathrm{d}o_{\partial C(t)}} = f(t, .) - \kappa(t)$$

exists if the correction function $\kappa(t)$ equals $r \cdot t$. The solution $C(.)$ is given by $C(t) = r\mathbb{B} + (r \cdot t, 0, \ldots, 0)$ and $\kappa(.)$ fulfills of course for any $t \in [0, \infty)$ the equation

$$\int_{\partial C(t)} f(x, t) - \kappa(t)\,\mathrm{d}o_{\partial C(t)} = \int_{(rt,0,\ldots,0)+r\partial\mathbb{B}} (x_1 - r \cdot t)\,\mathrm{d}o_{(rt,0,\ldots,0)+r\partial\mathbb{B}} = 0.$$

Proof of Theorem 1. Without loss of generality, there exist constants $r, R > 0$ such that $r\mathbb{B} \subset C_0$, $C_0 + r\mathbb{B} \subset U$ and $U \subset R\mathbb{B}$. We note that by the compactness of U we get $g_{\min} := \min_{x\in U} g(x) > 0$. Let ρ be such that:

$$\frac{1}{\rho} := \max\left(l,\ \frac{1}{\hat{\rho}},\ \frac{2R}{r^2}\left[\|f\| + \frac{\|f\| \cdot \|g\| \cdot \mathrm{O}(U)}{g_{\min} \cdot \mathrm{O}(\frac{1}{2}r\mathbb{B})}\right]\right).$$

Let $s(n) := \rho/2^n$ and let us define set-valued functions $C^n(t)$ and (for notational convenience) functions κ^n by induction on the length of the domain as follows:

We start at the point 0. Let $\kappa^n(0)$ be defined by

$$\kappa^n(0) := \frac{V'(C_0, f(0,.), \mu)}{V'(C_0, 1, \mu)},$$

let $\tilde{f}^n(0,.) := f(0,.) - \kappa^n(0)$ and let $C^n\colon [0, s(n)] \mapsto \mathcal{C}$ be defined by

$$C^n(t) := C_{\tilde{f}^n(0,.)}(t),$$

where $C_{\tilde{f}^n(0,.)}(t)$ denotes the function defined in Lemma 1 with $f(.)$ replaced by $\tilde{f}^n(0,.)$.

Now we proceed by induction on m:

Assume that $C^n(t)$ has already been defined for $t \in [0, s(n)\cdot m]$ for some $m \in \{1, \ldots, 2^n - 1\}$ and that

$$(1)\quad \left(\frac{1}{2} + \frac{2^n - m}{2^{n+1}}\right) r\mathbb{B} \subseteq C^n(s(n)\cdot m) \subseteq C_0 + \frac{m}{2^{n+1}} r\mathbb{B}.$$

We define $\kappa^n(s(n)\cdot m)$ by

$$\kappa^n(s(n)\cdot m) := \frac{V'(C^n(s(n)\cdot m), f(s(n)\cdot m), .), \mu)}{V'(C^n(s(n)\cdot m, 1, \mu)}.$$

For notational convenience we set $\tilde{f}^n(s(n)\cdot m, .) := f(s(n)\cdot m, .) - \kappa^n(s(n)\cdot m)$ and define $C^n\colon (s(n)\cdot m, s(n)\cdot(m+1)] \mapsto \mathcal{C}(\mathbb{E}^k)$ by

$$C^n(t) := C^n(s(n)\cdot m)_{\tilde{f}^n(s(n)\cdot m,.)}(t).$$

Here $C^n(s(n)\cdot m)$ denotes the function $C^n(.)$ at the point $(s(n)\cdot m)$ where it has already been defined by induction hypothesis. $C^n(s(n)\cdot m)_{\tilde{f}^n(s(n)\cdot m,.)}(t)$ denotes the function defined in Lemma 1 with $f(.)$ replaced by $\tilde{f}^n(s(n)\cdot m, .)$ and C_0 replaced by $C^n(s(n)\cdot m)$ at the point $(t - s(n)\cdot m)$. To be more precise we define C^n on the interval $(s(n)\cdot m, s(n)\cdot(m+1)]$ by:

$$C^n(t) = [C^n(s(n)\cdot m) \cup C_+^{n,m}(t)] \setminus C_-^{n,m}(t)$$

with

$$C_+^{n,m}(t) := \bigcup_{\{x\in\partial C(s(n)\cdot m) \mid \tilde{f}^n(s(n)\cdot m)\geqslant 0\}} (x + (t - s(n)\cdot m)\mathbb{B})$$

and

$$C_-^{n,m}(t) := \bigcup_{\{x\in\partial C(s(n)\cdot m) \mid \tilde{f}^n(s(n)\cdot m)< 0\}} (x + (t - s(n)\cdot m)\mathbb{B}^\circ)$$

Since $\frac{1}{2}r\mathbb{B} \subseteq C^n(s(n)\cdot m) \subseteq U$ we obtain from the definition of κ^n

$$|\kappa^n| \leqslant \frac{\|f\|\cdot\|g\|\cdot \mathrm{O}(U)}{g_{\min}\cdot \mathrm{O}(\frac{1}{2}r\mathbb{B})}.$$

We thus get

$$|\tilde{f}^n(s(n)\cdot m,.)| \leqslant \|f\| + \frac{\|f\|\cdot\|g\|\cdot \mathrm{O}(U)}{g_{\min}\cdot \mathrm{O}(\frac{1}{2}r\mathbb{B})}$$

and thus have by (1) and the definition of $s(n)$, ρ and $C^n(.)$

$$\begin{aligned}
&\left(\frac{1}{2}+\frac{2^n-(m+1)}{2^{n+1}}\right)r\mathbb{B}\\
&\subseteq \left(\frac{1}{2}+\frac{2^n-m}{2^{n+1}}\right)r\mathbb{B} \ominus \frac{r^2}{2R\cdot 2^n}\mathbb{B}\\
&\subseteq \left(\frac{1}{2}+\frac{2^n-m}{2^{n+1}}\right)r\mathbb{B} \ominus \frac{\rho}{2^n}\|\tilde{f}^n(s(n)\cdot m,.)\|\mathbb{B}\\
&\subseteq C^n(s(n)\cdot m) \ominus s(n)\cdot\|\tilde{f}^n(s(n)\cdot m,.)\|\mathbb{B}\\
&\subseteq C^n(s(n)\cdot(m+1)) \subseteq C^n(s(n)\cdot m) + s(n)\|\tilde{f}^n(s(n)\cdot m,.)\|\mathbb{B}\\
&\subseteq C_0 + \frac{m}{2^{n+1}}r\mathbb{B} + \frac{\rho}{2^n}\|\tilde{f}^n(s(n)\cdot m,.)\|\mathbb{B}\\
&\subseteq C_0 + \frac{m}{2^{n+1}}r\mathbb{B} + \frac{r^2}{2R\cdot 2^n}\mathbb{B} \subseteq C_0 + \frac{m+1}{2^{n+1}}r\mathbb{B},
\end{aligned}$$

which shows that (1) also holds for $m+1$ as long as $m < 2^n$. Thus

$$\frac{1}{2}r\mathbb{B} \subseteq \left(\frac{1}{2}+\frac{2^n-m}{2^{n+1}}\right)r\mathbb{B} \subseteq C^n(s(n)\cdot m) \subseteq C_0 + \frac{m}{2^{n+1}}r\mathbb{B} \subseteq U$$

holds for all $m < 2^n$ and we see that the induction process does not terminate before $m = 2^n$. This shows that the set-valued functions $C^n(t)$ are $\forall n \in \mathbb{N}$ defined on the whole interval $[0, \rho]$.

By the definition of $\kappa^n(.)$ and $\tilde{f}^n(.,.)$, by Lemma A.1 of the Appendix and by the Lipschitz continuity of $f(.,.)$, we obtain that the functions $\tilde{f}^n(.,.)$ are uniformly Lipschitz continuous with respect to some Lipschitz constant $\tilde{l}$; i.e. we have for $m_1, m_2 \in \{0,\dots,2^n\}$ and $x_1, x_2 \in U$

$$(2)\quad |\tilde{f}^n(s(n)\cdot m_1, x_1) - \tilde{f}^n(s(n)\cdot m_2, x_2)| \leqslant \tilde{l}(|s(n)\cdot(m_1-m_2)| + \|x_1-x_2\|).$$

Applying Proposition A.5 of the Appendix we see that the functions $C^n\colon [0,\rho) \mapsto \mathcal{C}$ are equicontinuous with respect to the Hausdorff metric and therefore by Ascolis theorem there exists a subsequence $C^{n_j}(t)$ which converges uniformly with respect to the Hausdorff metric to a function $C(t)$. It remains to show that $C(t)$ is weakly μ-differentiable and that the derivative is given by

$$\mathrm{d}\dot{C}^{\mu}(t) = (f(t,.)-\kappa(t))g(.)\,\mathrm{d}o_{C(t)}.$$

So let $t \in [0, \rho)$, $h \in [0, \rho - t)$ and let $m(j, t)$ and $m(j, t + h)$ be sequences of natural numbers with $\lim_{j\to\infty} m(j, t) \cdot s(n_j) = t$ and $\lim_{j\to\infty} m(j, t+h) \cdot s(n_j) = t + h$. Let ψ be an arbitrary real valued continuous function on $\mathbb{E}^k$. Let $\delta > 0$ be arbitrary and let $j \in \mathbb{N}$ be such that if we set (for simplicity of notation)

$$t(\delta) := s(n_j) \cdot m(j, t) \quad \text{and} \quad h(\delta) := s(n_j) \cdot (m(j, t + h) - m(j, t))$$

we have

$$(3) \quad \int \psi |\mathbb{1}_{C^{n_j}(t(\delta))} - \mathbb{1}_{C(t)}| \, \mathrm{d}\mu, \ \int \psi |\mathbb{1}_{C^{n_j}(t(\delta)+h(\delta))} - \mathbb{1}_{C(t+h)}| \, \mathrm{d}\mu \leqslant \delta \cdot h,$$

$$(4) \quad |h(\delta)| \leqslant 2h$$

and

$$(5) \quad |V'(C(t(\delta)), [f(t(\delta), .) - \kappa^{n_j}(t(\delta))], \psi(.)g(.)) - \\ -V'(C(t), [f(t, .) - \kappa(t)], \psi(.)g(.))| \leqslant \varepsilon_\psi(h),$$

where $\varepsilon_\psi(.)$ denotes the function of Lemma 2. (That there exists a δ such that (5) holds follows since Lemma A.2 of the Appendix, the definition of $\kappa^{n_j}(t(\delta))$, $\kappa(t)$ and the fact that $C^{n_j}(t(\delta))$ converges to $C(t)$ with respect to the Hausdorff metric imply that $\kappa^{n_j}(t(\delta))$ converges to $\kappa(t)$.)

By Lemma 3 and since by (2) the function $\tilde{f}^{n_j}$ is Lipschitz continuous with Lipschitz constant $\tilde{l}$ we have

$$(6) \quad d(C^{n_j}(t(\delta) + h(\delta)), C^{n_j}(t(\delta))_{\tilde{f}^{n_j}(t(\delta),.)}(h(\delta))) \leqslant (h(\delta))^2 \tilde{l}(\|\tilde{f}^{n_j}\| + 1)\frac{2R}{r}$$

and by Lemma 2

$$(7) \quad \left| \int \psi \frac{\mathbb{1}_{C^{n_j}(t(\delta))_{\tilde{f}^{n_j}(t(\delta),.)}(h(\delta))} - \mathbb{1}_{C^{n_j}(t(\delta))}}{h} \, \mathrm{d}\mu - \right. \\ \left. - \int_{\partial C^{n_j}(t(\delta))} \psi(.)g(.)\tilde{f}^{n_j}(t(\delta), .) \, \mathrm{d}o_{\partial C^{n_j}(t(\delta))} \right| \leqslant \varepsilon_\psi(h).$$

Then

$$\left| \int \psi \left(\frac{\mathrm{d}\mu|_{C(t+h)} - \mathrm{d}\mu|_{C(t)}}{h} - \mathrm{d}\dot{C}^\mu(t) \right) \right|$$
$$= \left| \int \psi \left(\frac{\mathbb{1}_{C(t+h)} - \mathbb{1}_{C(t)}}{h} \mathrm{d}\mu - \mathrm{d}\dot{C}^\mu(t) \right) \right|$$
$$\leqslant \left| \int \psi \left(\frac{\mathbb{1}_{C^{n_j}(t(\delta)+h(\delta))} - \mathbb{1}_{C^{n_j}(t(\delta))}}{h} \mathrm{d}\mu - \mathrm{d}\dot{C}^\mu(t) \right) \right| + 2\delta$$
$$\leqslant \left| \int \psi \left(\frac{\mathbb{1}_{C^{n_j}(t(\delta))_{\tilde{f}^{n_j}(t(\delta),.)}(h(\delta))} - \mathbb{1}_{C^{n_j}(t(\delta))}}{h} \mathrm{d}\mu - \mathrm{d}\dot{C}^\mu(t) \right) \right| +$$

$$
\begin{aligned}
&+2\delta + \frac{(h(\delta))^2}{h}\tilde{l}(\|\tilde{f}^{n_j}\|+1)\frac{2R}{r}\cdot\|\psi\|\cdot\|g\|\cdot 2\mathrm{O}(R\mathbb{B})\\
\leqslant\ &\left|\int_{\partial C^{n_j}(t(\delta))}\psi(.)g(.)\tilde{f}^{n_j}(t(\delta),.)\,\mathrm{d}o_{\partial C^{n_j}(t(\delta))} - \int\psi\,\mathrm{d}\dot{C}^{\mu}(t)\right| +\\
&+2\delta + 4h\tilde{l}(\|\tilde{f}^{n_j}\|+1)\frac{2R}{r}\cdot\|\psi\|\cdot\|g\|\cdot 2\mathrm{O}(R\mathbb{B}) + \varepsilon_{\psi}(h)\\
=\ &|V'(C(t(\delta)), f(t(\delta),.) - \kappa^n(t(\delta)), \psi(.)g(.)) -\\
&- V'(C(t), f(t,.) - \kappa(t), \psi(.)g(.))| + 2\delta +\\
&+ 4h\tilde{l}(\|\tilde{f}^{n_j}\|+1)\frac{2R}{r}\cdot\|\psi\|\cdot\|g\|\cdot 2\mathrm{O}(R\mathbb{B}) + \varepsilon_{\psi}(h)\\
\leqslant\ &2\delta + 4h\tilde{l}(\|\tilde{f}^{n_j}\|+1)\frac{2R}{r}\cdot\|\psi\|\cdot\|g\|\cdot 2\mathrm{O}(R\mathbb{B}) + 2\varepsilon_{\psi}(h),
\end{aligned}
$$

where the first inequality follows from (3), the second from (6) and the third inequality from (7) and (4). The quality sign in the penultimate step of the calculation follows from the definitions of $V'(.,.,.)$ and Lemma 2. The last inequality follows from (5).

Thus since h and δ can be chosen arbitrarily small

$$\mathrm{d}(\dot{C}^{\mu}(t)) = (f(t,.) - \kappa(t))g(.)\,\mathrm{d}o_{C(t)}$$

has been shown. It remains only to show that $\mu(C(t)) = \mu(C_0)$ to complete the proof. Since $\mu(C(0)) = \mu(C_0)$ it is sufficient to show that $\mathrm{d}\mu(C(t))/\mathrm{d}t = 0$, which we are going to do now:

$$
\begin{aligned}
0\ &\leqslant\ \lim_{h\mapsto 0}\left|\frac{\mu(C(t+h)) - \mu(C(t))}{h}\right|\\
&=\ |V'(C(t), f(t,.) - \kappa(t), g(.))|\\
&=\ \left|\int_{\partial C(t)} f(t,.)g(.)\,\mathrm{d}o_{\partial C(t)} - \int_{\partial C(t)} g(.)\kappa(t)\,\mathrm{d}_{o_{\partial C(t)}}\right| = 0,
\end{aligned}
$$

which follows from the definition of $\kappa(.)$ and $V'(.,.,.)$. Thus, the theorem is proved. □

THEOREM 1′. *Let $C_0 \in (\mathcal{C}, d)$ and let U be a compact convex neighborhood of C_0. Let $f \in \mathcal{G}([0,\hat{\rho}]\times U, l)$ be such that for any t the function $f(t,.)$ is concave and let $\{\mu(t)\}_{t\in[0,\hat{\rho}]}$ be a weak differentiable function of positive measures with $\mathrm{d}(\mu(t)) := g(t,.)\,\mathrm{d}\lambda$ where $g(t,.) \in \mathcal{G}(U,l)$ whose derivative $\mu'(.)$ possesses for any t a Lipschitz continuous density $h(t,.) \in \mathcal{G}(U,l)$ with respect to Lebesgue measure. Let further α: $[0,\hat{\rho}] \mapsto \mathbb{R}^+$ be a function with Lipschitz continuous derivative such that for any t there exists a bounded set $D(t) \subset U$ with $\mu_t(D(t)) = \alpha(t)$ and $\mu_0(C_0) = \alpha(0)$. Then there exists $\rho \in (0,\hat{\rho}]$ and a set-valued mapping C: $[0,\rho) \mapsto \mathcal{C}$ with convex compact values such that*

$$C(0) = C_0 \ \text{ and } \ \mu_t(C(t)) = \alpha(t)$$

and

$$\frac{\mathrm{d}\dot{C}^{\mu(t)}(t)}{\mathrm{d}o_{C(t)}} = [f(t,.) - \kappa(t) + \beta(t) - \gamma(t)]g(t,.),$$

where $\beta(t)$, $\gamma(t)$ *and* $\kappa(t)$ *are such that*

$$\int_{\partial C(t)} g(t,.)\kappa(t)\,\mathrm{d}o = \int_{C(t)} f(t,.)\cdot g(t,.)\,\mathrm{d}o,$$

$$\int_{\partial C(t)} g(t,.)\beta(t)\,\mathrm{d}o = \alpha'(t) \quad \text{and} \quad \int_{\partial C(t)} g(t,.)\gamma(t)\,\mathrm{d}o = \mu'(t)(C(t)).$$

Appendix

Remark A.1. Note that the surface area $\mathrm{O}(C)$ of a convex set $C \subset \mathbb{E}^k$ may be defined as $\mathrm{O}(C) = kv(C, \ldots, C, \mathbb{B})$, where v: $\mathcal{C}^k \mapsto \mathbb{R}$ denotes the (k-dimensional) mixed volume. (For the definition of mixed volumes see, for example, [11, 17].) In the following we make use of the following properties of mixed volumes (for their proof see [11, 17]):

(i) $v(C, \ldots, C) = V(C)$,
(ii) $v(C_1, \ldots, C_k) = v(C_{i_1}, \ldots, C_{i_k})$ where $(i_1, \ldots, i_k)$ is an arbitrary permutation of $(1, \ldots, k)$,
(iii) Mixed volumes are linear in the first component in the sense that for $\alpha, \alpha' \geqslant 0$

$$v(\alpha C_1 + \alpha' C_1', C_2, \ldots, C_k) = \alpha v(C_1, C_2, \ldots, C_k) + \alpha' v(C_1', C_2, \ldots, C_k)$$

and are thus multilinear because of (ii).
(iv) Mixed volumes are monotonic; i.e.

$$C_1 \subseteq D_1, \ldots, C_k \subseteq D_k \Rightarrow v(C_1, \ldots, C_k) \leqslant v(D_1, \ldots, D_k).$$

PROPOSITION A.1. *Let* $C \in \mathcal{C}$ *and let* h: $[0, \infty) \mapsto [0, \infty)$ *be defined by* $h(\xi) := \mathrm{O}(C + \xi r\mathbb{B})$*, then h is a monotone increasing convex function.*

Proof. By use of mixed volumes we have

$$h(\xi) = kv(C + \xi r\mathbb{B}, \ldots, C + \xi r\mathbb{B}, \mathbb{B})$$

so that we get the monotonicity of $h(\xi)$ by the monotonicity of mixed volumes. To prove the convexity of h it is clearly sufficient to show that the function

$$g(\xi) := \lim_{\varepsilon \to 0+} \frac{h(\xi + \varepsilon) - h(\xi)}{\varepsilon}$$

is monotonic in ξ. But

$$g(\xi) = \lim_{\varepsilon\to 0+}\left[\frac{kv(C+(\xi+\varepsilon)r\mathbb{B},\dots,C+(\xi+\varepsilon)r\mathbb{B},\mathbb{B})}{\varepsilon} - \right.$$
$$\left. - \frac{kv(C+\xi r\mathbb{B},\dots,C+\xi r\mathbb{B},\mathbb{B})}{\varepsilon}\right].$$

and by properties (ii) and (iii) of mixed volumes the expression on the right equals

$$\lim_{\varepsilon\to 0+}\frac{k\sum_{(i_1,\dots,i_{k-1})\in I} v(D_{i_1},\dots,D_{i_{k-1}},\mathbb{B})}{\varepsilon},$$

where I denotes the set of 0–1 tuples of length $k-1$ with at least one entry equal to 1, $D_0 := C+\xi r\mathbb{B}$ and $D_1 := \varepsilon\mathbb{B}$. By application of (ii) and (iii) and calculation of the limit we finally get

$$g(\xi) := k(k-1)v(C+\xi r\mathbb{B},\dots,C+\xi r\mathbb{B},\mathbb{B},\mathbb{B})$$

and thus by monotonicity of mixed volumes we get monotonicity of g and convexity of h. □

Remark A.2. From Proposition A.1 we immediately infer that $\xi \in [0,1]$ implies

$$\mathrm{O}(C+\xi r'\mathbb{B}) - \mathrm{O}(C) \leqslant \xi(\mathrm{O}(C+r'\mathbb{B}) - \mathrm{O}(C)).$$

PROPOSITION A.2. *Let $C, C' \in \mathcal{C}(\mathbb{E}^k)$ and $C \subseteq C'$ then*

(i) $\mathrm{O}(C) \leqslant \mathrm{O}(C')$ (*monotonicity of the surface area*)

and

(ii) $\mathrm{O}(C+r\mathbb{B}) - \mathrm{O}(C) \leqslant \mathrm{O}(C'+r\mathbb{B}) - \mathrm{O}(C')$
(*monotonicity of the increase of the surface area*).

Proof. That $C \subseteq C' \Rightarrow \mathrm{O}(C) \leqslant \mathrm{O}(C')$ is clear from the definition of the surface area by mixed volumes and the definition of mixed volumes. The second assertion is proved as follows:

$$\begin{aligned}
&\mathrm{O}(C+r\mathbb{B}) - \mathrm{O}(C)\\
&= kv(C+r\mathbb{B},\dots,C+r\mathbb{B},\mathbb{B}) - kv(C,\dots,C,\mathbb{B})\\
&\quad = k\sum_{(i_1,\dots,i_{k-1})\in I} v(Q_{i_1},\dots,Q_{i_{k-1}},\mathbb{B})\\
&\quad \leqslant k\sum_{(i_1,\dots,i_{k-1})\in I} v(R_{i_1},\dots,R_{i_{k-1}},\mathbb{B})\\
&\quad = kv(C'+r\mathbb{B},\dots,C'+r\mathbb{B},\mathbb{B}) - kv(C',\dots,C',\mathbb{B})\\
&\quad = \mathrm{O}(C'+r\mathbb{B}) - \mathrm{O}(C),
\end{aligned}$$

where I denotes the set of 0–1 tuples of length $k-1$ with at least one entry equal to 1, $Q_0 := C$, $R_0 := C'$ and $R_1 = Q_1 := r\mathbb{B}$. □

PROPOSITION A.3. *Let $C, C' \in \mathcal{C}(\mathbb{E}^k)$ and $C \subseteq C'$, then*

$$V(C + r\mathbb{B} \setminus C) \leqslant V(C' + r\mathbb{B} \setminus C).$$

Proof. This follows by integration since by assertion (i) of Proposition A.2 we have $\mathrm{O}(C + t\mathbb{B}) \leqslant \mathrm{O}(C' + t\mathbb{B})$ and thus

$$V(C + r\mathbb{B} \setminus C) = \int_0^r \mathrm{O}(C + t\mathbb{B})\,\mathrm{d}t \leqslant \int_0^r \mathrm{O}(C' + t\mathbb{B}) = V(C + r\mathbb{B} \setminus \mathbb{B}). \quad \square$$

PROPOSITION A.4. *Let $C_1, C_2 \in \mathcal{C}$ be such that $r\mathbb{B} \subset C_1, C_2 \subset R\mathbb{B}$. Then $d(C_i, C_1 \cap C_2) \leqslant d(C_1, C_2)R/r$.*

PROPOSITION A.5. *Let $C_0 \in \mathcal{C}$ be such that $r\mathbb{B} \subset C_0 \subset R\mathbb{B}$ and $f \in \mathcal{F}(C_0, l)$ then for all $s \in [0, \frac{1}{l}]$ we have $d(C_0, C_f(s)) \leqslant sR/r\|f\|$.*

PROPOSITION A.6. *Let (X, μ) and (Y, ν) be two measure spaces, Ξ: $X \mapsto Y$ a measure decreasing function and let h: $X \cup Y \mapsto \mathbb{R}$ be an integrable function. Then we have for arbitrary $\nu' \in \Xi^{-1}(\nu)$*

$$\left|\int_X h\,\mathrm{d}\mu - \int_Y h\,\mathrm{d}\nu\right| \leqslant \|h\|(\mu(X) - \nu(Y)) + \left|\int_X h\,\mathrm{d}\nu' - \int_Y h\,\mathrm{d}\nu\right|.$$

LEMMA A.1. *Let $C \in (\mathcal{C}, d)$ and U be a bounded convex neighborhood of C in $\mathbb{E}^k$, then there exists a neighborhood $\mathcal{U}(C) \subset (\mathcal{C}, d)$ such that*

$$V'(C', f, g) = \int_{\partial C'} f \cdot g\,\mathrm{d}o_{\partial C'}$$

fulfills a Lipschitz condition on $(\mathcal{U}(C), d) \times B_{\mathcal{F}} \times B_{\mathcal{G}}$, where $B_{\mathcal{F}} \subset \mathcal{F}(U, l)$ and $B_{\mathcal{G}} \subset \mathcal{G}(U, l)$ are bounded with respect to the supnorm.

Proof. Let us choose $\mathcal{U}(C)$ such that $\bigcap \mathcal{U}(C)$ contains a ball B of radius r and $\bigcup \mathcal{U}(C) \subset U$. Let now $(C_1, f_1, g_1), (C_2, f_2, g_2) \in (\mathcal{U}(C), d) \times \mathcal{F}(U, l) \times \mathcal{G}(U, l)$ be given. Note, that the functions $h_i := f_i \cdot g_i$ are also Lipschitz continuous with Lipschitz constant $l' := l(\|f_i\| + \|g_i\|)$. Since the intersection of C_1 and C_2 is not empty we have $C_i \subset C_1 \cap C_2 + d(C_1 \cap C_2, C_i)\mathbb{B}$. This by an application of Proposition A.2 gives

$$\begin{aligned}
&\mathrm{O}(C_i) - \mathrm{O}(C_1 \cap C_2)\\
&\quad\leqslant \mathrm{O}(C_1 \cap C_2 + \mathrm{d}(C_1 \cap C_2, C_i)\mathbb{B}) - \mathrm{O}(C_1 \cap C_2)\\
&\quad\leqslant \mathrm{O}(U + d(C_1 \cap C_2, C_i)\mathbb{B}) - \mathrm{O}(U),
\end{aligned}$$

where the first inequality follows from Proposition A.2(i) and the second from Proposition A.2(ii). Therefrom we get by an application of Remark A.2 with

$$\xi = \frac{\mathrm{d}(C_1 \cap C_2, C_i)}{\mathrm{diam}(\mathcal{U}(C))}, \qquad r' = \mathrm{diam}(\mathcal{U}(C)) \quad \text{and} \quad C = U.$$

$$\text{(1)}\quad \mathrm{O}(C_i) - \mathrm{O}(C_1 \cap C_2) \leqslant \frac{\mathrm{d}(C_1 \cap C_2, C_i)}{\mathrm{diam}(\mathcal{U}(C))}(\mathrm{O}(U + \mathrm{diam}(\mathcal{U}(C))\mathbb{B}) - \mathrm{O}(U)).$$

Since Proposition A.4 gives

$$\text{(2)}\quad d(C_1 \cap C_2, C_1), d(C_1 \cap C_2, C_2) \leqslant d(C_1, C_2)\frac{\mathrm{diam}(\mathcal{U}(C))}{r}$$

we get from (1)

$$\text{(3)}\quad \mathrm{O}(C_i) - \mathrm{O}(C_1 \cap C_2) \leqslant \frac{d(C_1, C_2)}{r}(\mathrm{O}(U + \mathrm{diam}(\mathcal{U}(C))\mathbb{B}) - \mathrm{O}(U)).$$

Let pr_i denote the orthogonal projection of ∂C_i onto $\partial(C_1 \cap C_2)$ and let o_i be an arbitrary measure on ∂C_i with $pr_i(o_i) = o$ where o denotes the surface area measure of $C_1 \cap C_2$. We note that the projection decreases the surface area measure. Thus

$$\begin{aligned}
&|V'(C_1, f_2, g_2) - V'(C_2, f_2, g_2)| \\
&\quad = \left| \int_{\partial C_1} h_2 \,\mathrm{d}o_{\partial C_1} - \int_{\partial C_2} h_2 \,\mathrm{d}o_{\partial C_2} \right| \\
&\quad \leqslant \sum_{i \in \{1,2\}} \left| \int_{\partial C_i} h_2 \,\mathrm{d}o_{\partial C_i} - \int_{\partial(C_1 \cap C_2)} h_2 \,\mathrm{d}o \right| \\
&\quad \leqslant \sum_{i \in \{1,2\}} \|h_2\|(\mathrm{O}(C_i) - \mathrm{O}(C_1 \cap C_2)) + \left| \int_{\partial C_i} h_2 \,\mathrm{d}o_i - \int_{\partial(C_1 \cap C_2)} h_2 \,\mathrm{d}o \right| \\
&\quad \leqslant 2\|h_2\| \frac{\mathrm{d}(C_1, C_2)}{r}(\mathrm{O}(U + \mathrm{diam}(\mathcal{U}(C))\mathbb{B}) - \mathrm{O}(U)) + \\
&\qquad + \sum_i \int_{\partial C_i} l' \mathrm{d}(C_i, C_1 \cap C_2) \,\mathrm{d}o_i + \left| \int_{\partial(C_1 \cap C_2)} h_2 \,\mathrm{d}o - \int_{\partial(C_1 \cap C_2)} h_2 \,\mathrm{d}o \right| \\
&\quad \leqslant 2\|h_2\| d(C_1, C_2) \frac{(\mathrm{O}(U + \mathrm{diam}(\mathcal{U}(C))\mathbb{B}) - \mathrm{O}(U))}{r} + \\
&\qquad + 2l' d(C_1, C_2) \frac{\mathrm{diam}(\mathcal{U}(C))}{r} \mathrm{O}(U),
\end{aligned}$$

where the second inequality follows from Proposition A.6, the third from (3) and the fact that h fulfills a Lipschitz condition with Lipschitz constant l' and the fourth inequality from (2) and monotonicity of the surface area (Proposition A.2(i)). (Note that the last term in the penultimate expression equals of course 0.)

So we see that V' fulfills a Lipschitz condition in the first variable. That V' fulfills a Lipschitz condition in the second and third variable is clear since using monotonicity of the surface area (proved in Proposition A.2 of the Appendix) we get

$$
\begin{aligned}
&|V'(C_1, f_1, g_1) - V'(C_1, f_2, g_2)| \\
&\quad = \int_{\partial C_1} |h_1 - h_2| \, \mathrm{d}o \leqslant \|h_1 - h_2\| \int_{\partial U} \mathrm{d}o \\
&\quad \leqslant \mathrm{O}(U) \sup_{f \in B_{\mathcal{F}}} \|f\| \cdot \|g_1 - g_2\| + \mathrm{O}(U) \sup_{g \in B_{\mathcal{G}}} \|g\| \cdot \|f_1 - f_2\|.
\end{aligned}
$$

So we finally get

$$
\begin{aligned}
&|V'(C_1, f_1, g_1) - V'(C_2, f_2, g_2)| \\
&\quad \leqslant L_1 \cdot [d(C_1, C_2) + \|f_1 - f_2\| + \|g_1 - g_2\|]
\end{aligned}
$$

with L_1 depending only on $(U, \operatorname{diam}(\mathcal{U}(C)), r, B_{\mathcal{F}}, B_{\mathcal{G}})$.

Thus the lemma is proved. □

LEMMA A.2. *Let U be an open convex set, let $f \in \mathcal{F}(U, l)$ and let φ be an element of the space of continuous functions on $\mathbb{E}^k$ with compact support endowed with the sup-norm. Then $(C, f, \varphi) \mapsto V'(C, f, \varphi)$ is continuous in the respective topologies.*

Proof. The result is proved analogous to Lemma A.1. □

Sketch of the proof of Example 6. For any $n \in \mathbb{N}$ define set-valued functions $C^n(t)$ by induction as follows:

We start the induction procedure by defining $C(t)$ on the interval $[0, 1/2^n]$:

Let $C^n(0) := C_0$ and $r^n(0) = r_0$. Chose $\kappa^n(0)$ such that if we let $r^n(t) := r^n(0) + t \cdot \kappa^n(0)$ for $t \in (0, 1/2^n]$ and if we define the set-valued function $C^n(t)$ on $(0, 1/2^n]$ by $C^n(t) = r^n(t)\mathbb{B} + (\int_0^t r^n(s)\,\mathrm{d}s, \vec{0})$, then $r^n(1/2^n) > 0$ and $\mu(C^n(1/2^n)) = \mu(C_0)$. (Note that there always exists a $\kappa^n(0)$ we can choose.)

Now we proceed by induction on the domain of $C^n(t)$:

Suppose that $C^n(t)$ has already been defined on $[0, m/2^n]$ for some $m \in \mathbb{N}$. We choose $\kappa^n(m/2^n)$ such that if we let

$$
r^n(t) := r^n\left(\frac{m}{2^n}\right) + (t - m/2^n) \cdot \kappa^n\left(\frac{m}{2^n}\right) \quad \text{for } t \in (m/2^n, (m+1)/2^n]
$$

and we define $C^n(t)$ on $(m/2^n, (m+1)/2^n]$ by $C^n(t) = r^n(t)\mathbb{B} + (\int_0^t r^n(s)\,\mathrm{d}s, \vec{0})$, then $r^n((m+1)/2^n) > 0$ and $\mu(C^n((m+1)/2^n) = \mu(C_0)$. (Again there exists always a $\kappa^n((m+1)/2^n)$ we can choose.)

We see that for any $t \in [0, \infty)$ and any $n \in \mathbb{N}$ the set $C^n(t)$ is a ball. Further by the theorem of Ascoli one can show that a subsequence $\langle C^{n_l}(t) \rangle_{l \in \mathbb{N}}$ exists which converges with respect to the Hausdorff metric to some set-valued function $C(t)$. The values of this function $C(t)$ are thus balls. One can further prove that the function $C(t)$ is weakly differentiable from the right, that $\mu(C(t)) = \mu(C_0)$ and that $C(t)$ fulfills (4) and (5) for some suitable function $\kappa(t)$. □

Remark. This result can also be proved by showing that the local solutions provided by Theorem 1 are ball valued in case of Example 6 and that it is possible to glue such local solutions together to provide global ones. This will be done in another article.

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